CONVERGENCE OF CLOCK PROCESSES AND AGING IN METROPOLIS DYNAMICS OF A TRUNCATED REM

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ABSTRACT. We study the aging behavior of a truncated version of the Random Energy Model evolving under Metropolis dynamics. We prove that the natural time-time correlation function defined through the overlap function converges to an arcsine law distribution function, almost surely in the random environment and in the full range of time scales and temperatures for which such a result can be expected to hold. This establishes that the dynamics ages in the same way as Bouchaud's REM-like trap model, thus extending the universality class of the latter model. The proof relies on a clock process convergence result of a new type where the number of summands is itself a clock process. This reflects the fact that the exploration process of Metropolis dynamics is itself an aging process, governed by its own clock. Both clock processes are shown to converge to stable subordinators below certain critical lines in their time-scale and temperature domains, almost surely in the random environment.

1. INTRODUCTION AND MAIN RESULTS

As evidenced by an extensive body of experiments, glassy systems are never in equilibrium on laboratory time scales [12], [32]; instead, their dynamics become increasingly slower as time elapses. Termed *aging*, this pattern of behavior was most successfully accounted for, at a theoretical level, by Bouchaud's phenomenological trap models [11], [13]. These are effective dynamics that, reviving ideas of Goldstein *et al.* [28], model the long time behavior of spin glass dynamics in terms of thermally activated barrier crossing in a state space reduced to the configurations of lowest energy (see [12] for a review). Main examples of microscopic systems that trap models aim to describe are Glauber dynamics on state spaces $\{-1, 1\}^n$ reversible with respect to the Gibbs measures associated to random Hamiltonians of mean-field spin glasses, such as the Random Energy Model (REM) and p-spin SK models [19], [20]. The link between such dynamics and their associated trap models is, however, simply postulated.

When trying to establish this link rigorously, a main question that arises is what Glauber dynamics to choose. While classical choices are Metropolis [31] or Heath-Bath dynamics [27], most of the focus so far was on the so-called *Random Hopping* dynamics whose transition rates do not depend on the variation of energy along a given transition but only on the energy of its starting point [4], [5], [6], [2], [25], [14], [15], [8], [23]. Although physically unrealistic, the relative simplicity of this choice allowed important insights to be gained: a rigorous justification of the connection between the REM dynamics and trap models was given, first on times scales close to equilibrium [3, 4, 5], later also on shorter

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(but still exponential in n) time scales [6], and these results were partially extended to the p-spin SK models [2] on a sub-domain of times scales, albeit only in law with respect to the random environment and for $p \geq 3$. The SK model itself ($p = 2$) could be dealt with on time scales that are sub-exponential in n and again in law with respect to the random environment [8]. A variant of the so-called Bouchaud's asymmetric dynamics in which the asymmetry parameter tends to zero as $n \uparrow \infty$ is considered in [30] for the REM.

Beyond model-based analysis, a general aging mechanism was isolated that linked aging to the arcsine law for subordinators through the asymptotic behavior of a partial sum process called *clock process*. First implemented in [6] in the setting of Random Hopping dynamics this mechanism was revisited in [26] and [14] where, using a method developed by Durrett and Resnick [22] to prove functional limit theorems for dependent random variables, simple and robust criteria for convergence of clock processes to subordinators were given, suited for dealing with general Glauber dynamics. Applied to the Random Hopping dynamics of the REM [25] and p-spin SK models [14], [15], these criteria allowed to improve all earlier results, turning statements previously obtained in law into almost sure statements in the random environment.

In the present paper the approach of [26] is applied to Metropolis dynamics of the REM for which it was primarily intended, although only for a truncated version of the REM Hamiltonian. While the ultimate goal is of course to deal with the full REM, the truncated model does captures a number of features that are present in the activated dynamics of the full model, and enables us to clarify a number of issues on a problem for which nothing is known at a theoretical level and no computer simulations are available.

1.1. **The setting.** Before entering into the details, let us specify the model. Denote by $\mathcal{V}_n = \{-1, 1\}^n$ the n-dimensional discrete cube and let $(g(x), x \in \mathcal{V}_n)$ be a collection of independent standard Gaussian random variables, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will refer to these Gaussians as to the *random environment.* The Hamiltonian or energy function of the standard REM simply is the random funtion

$$H_n^{\mathsf{REM}}(x) \equiv \sqrt{n} g(x) , \quad x \in \mathcal{V}_n. \tag{1.1}$$

Given a sequence $u_n > 0$ (our *truncation level*) the *truncated* REM Hamiltonian then is

$$H_n(x) \equiv \begin{cases} \sqrt{n} g(x), & \text{if } \; g(x) \leq -u_n, \\ 0, & \text{else;} \end{cases} , \quad x \in \mathcal{V}_n. \tag{1.2}$$

Here we follow the physical convention that the configurations of minimal energy are the most stable ones, that is to say, Gibbs measure at inverse temperature $\beta > 0$ is defined as

$$G_{\beta,n}(x) = e^{-\beta H_n(x)} / (\textstyle\sum_{x \in \mathcal{V}_n} e^{-\beta H_n(x)}) , \quad x \in \mathcal{V}_n \,. \tag{1.3}$$

We are interested in the single spin-flip continuous time Metropolis dynamics for this model. This is a Markov jump process $(X_n(t), t > 0)$ on $\mathcal{V}_n$ that is usually defined through its jump rates, given by

$$\lambda_n(x, y) = \begin{cases} \frac{1}{n} e^{-\beta [H_n(y) - H_n(x)]^+}, & \text{if } (x, y) \in \mathcal{E}_n, \\ 0, & \text{else;} \end{cases} \tag{1.4}$$

where $a+ = \max\{a, 0\}$, $\mathcal{E}_n = \{(x, y) \in \mathcal{V}_n \times \mathcal{V}_n \, : \, \mathrm{dist}(x, y) = 1\}$ is the set of edges of $\mathcal{V}_n$, and $\mathrm{dist}(x, x') \equiv \frac{1}{2} \sum_{i=1}^{n} |x_i - x'_i|$ is the graph distance on $\mathcal{V}_n$.

Equivalently, X_n can be defined as a time change of its *jump chain*, namely, the discrete time chain, J_n, that describes the trajectories of X_n, through the relation

$$X_n(t) = J(\widetilde{S}_n^{\leftarrow}(t)), \quad t \geq 0, \tag{1.5}$$

where $\widetilde{S}_n^{\leftarrow}$ denotes the generalized right continuous inverse of $\widetilde{S}_n$, and $\widetilde{S}_n$, the so-called *clock process*, is the partial sum process that records the total time spent by X_n along the trajectories of J_n. Spelling out these objects explicitly, the jump chain is the Markov chain $(J_n(i), i \in \mathbb{N})$ on $\mathcal{V}_n$ with one-step transition probabilities

$$p_n(x,y) = \frac{e^{-\beta[H_n(y)-H_n(x)]^+}}{\sum_{y:(x,y)\in\mathcal{E}_n} e^{-\beta[H_n(y)-H_n(x)]^+}}\,, \quad \text{if } (x,y) \in \mathcal{E}_n\,, \tag{1.6}$$

and $p_n(x,y) = 0$ otherwise, and the clock process is given by

$$\widetilde{S}_n(k) = \sum_{i=0}^{k-1} \lambda_n^{-1}(J_n(i))e_{n,i}\,, \quad k \geq 1, \tag{1.7}$$

where $(e_{n,i}\,, n \in \mathbb{N}, i \in \mathbb{N})$ is a collection of independent mean one exponential random variables, independent of J_n, and the $\lambda_n(\cdot)$'s are the classical holding time parameters

$$\lambda_n(x) \equiv \frac{1}{n} \sum_{y:(x,y)\in\mathcal{E}_n} e^{-\beta[H_n(y)-H_n(x)]^+}\,, \quad \forall x \in \mathcal{V}_n. \tag{1.8}$$

In the clock process-based aging mechanism, one aims to infer knowledge of the aging behavior of X_n as $n \uparrow \infty$ from the asymptotic behavior of the properly rescaled clock process, using relation (1.5). To formulate this more precisely let $K_n(t)$ be a nondecreasing right continuous function with range $\{0, 1, 2, \dots\}$ and let c_n be a nondecreasing sequence. Both $K_n(t)$ and c_n are time scales. Consider the re-scaled clock process

$$S_n(t) = c_n^{-1}\widetilde{S}_n(K_n(t))\,, \quad t \geq 0. \tag{1.9}$$

This is a doubly stochastic object: on the one hand, for each fixed realization of the random environment (that is, of the random Hamiltonian H_n), S_n is a partial sum process with increasing paths that increase only by jumps and whose increments depend on the $J_n(i)$'s and the $e_{n,i}$'s; on the other hand, both the $\lambda_n(\cdot)$'s and the law of J_n depend on the random environment. One then asks whether there exist time scales $K_n(t)$ and c_n that make S_n converge weakly, as $n \uparrow \infty$, as a sequence of random elements in Skorokhod's space $D([0,\infty))$, $\mathbb{P}$-almost surely in the random environment. Such a result will be useful for deriving aging information if it enables one to control the behavior of the two-time correlation functions that are used in theoretical physics to quantify this phenomenon, the natural choice in mean-field models being the two-time overlap function

$$\mathcal{C}_n(t,s) = n^{-1}\big(X_n(c_n t), X_n(c_n(t+s)\big) \tag{1.10}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^n$. Clearly, how successful this can be strongly depends on the topology in which weak convergence of S_n is obtained. *Normal* aging is then said to occur if, for some convergence mode,

$$\lim_{n\to\infty} \mathcal{C}_n(t,s) = \mathcal{C}_\infty(t/s) \tag{1.11}$$

for some non trivial function $\mathcal{C}_\infty$ (see [26] for more general aging behaviors). The key idea put forward in [6] is that if S_n converges to an α-stable subordinator with $\alpha \in (0,1)$ then (1.11) is nothing but a manifestation of the self-similarity of such subordinators, as captured by the Dynkin-Lamperti arcsine law Theorem.

For future reference, we denote $\mathcal{F}^J$ and $\mathcal{F}^X$, respectively, the σ-algebra generated by the variables J_n and X_n. We write P for the law of the process J_n, conditional on the σ-algebra $\mathcal{F}$, i.e. for fixed realizations of the random environment. Likewise we call $\mathcal{P}$ the law of X_n conditional on $\mathcal{F}$. If the initial distribution, say μ_n, has to be specified we write $\mathcal{P}_{\mu_n}$ and P_{μ_n}. Expectation with respect to $\mathbb{P}$, P, and $\mathcal{P}$ are denoted by $\mathbb{E}$, E, and $\mathcal{E}$, respectively.

1.2. **Results.** We must now specify the truncation level in (1.2). Given $c_\star > 0$, we let $u_n \equiv u_n(c_\star)$ be the sequence defined through

$$\mathbb{P}(g(x) \leq -u_n(c_\star)) = n^{-c_\star}. \tag{1.12}$$

Viewing the vertices of $\mathcal{V}_n$ as independently occupied with probability (1.12), one sees that this probability increases from 0 to 1 as $c_\star$ decreases from $+\infty$ to 0, and so, the set of occupied vertices evolves from the empty set to the entire $\mathcal{V}_n$. Set

$$\mathcal{V}_n^\star \equiv \{x \in \mathcal{V}_n \mid x \text{ is occupied}\} \setminus I_n^\star, \tag{1.13}$$

where $I_n^\star$ is the set of isolated occupied vertices, namely, $x \in I_n^\star$ if it is occupied but none of its n neighbors is. Our results are closely tied to the graph properties of this set. Let us only mention here that $c_\star$ will be chosen larger than three. This precludes the emergence of a giant connected component and guarantees that, $\mathbb{P}$- almost surely, the graph of $\mathcal{V}_n^\star$ is made of an exponentially large number ($\approx \mathcal{O}(2^n/n^{2c_\star-1})$) of small, disjoint connected components of size smaller than n. In explicit form, the sequence u_n obeys

$$u_n(c_\star) = \sqrt{2c_\star \log n} - \left(\tfrac{\log\log n + \log 4\pi}{2\sqrt{2c_\star \log n}} + \mathcal{O}\left(\tfrac{1}{\sqrt{\log n}}\right)\right). \tag{1.14}$$

Hence, the truncation only prunes energies such that $-H_n^{\mathsf{REM}}(x) \lesssim \sqrt{2c_\star n \log n}$, while activated aging typically involves energies of size $-H_n^{\mathsf{REM}}(x) \geq \gamma n$, $\gamma > 0$, that is to say, of the order of $\max_{x\in\mathcal{V}_n}(-H_n^{\mathsf{REM}}(x))$.

We are concerned with finding sequences c_n and K_n for which the rescaled clock process (1.9) converges for some (ideally, the smallest possible) $c_\star$. Note that in physical terms, c_n is the time scale on which the continuous time process X_n is observed, while $K_n(t)$ is the total number of steps the discrete time chain J_n takes during the period of observation. In all previously mentioned works on mean-field spin glasses (that is, the REM and p-spin SK models with $p \geq 3$) where convergence of (1.9) could be proved, this was on time scales of the form $c_n \sim \exp(\beta\gamma n)$, $\gamma > 0$. Furthermore, K_n invariably had to be chosen of the form $K_n(t) = \lfloor a_n t \rfloor$, where a_n is defined through $a_n \mathbb{P}(w_n(x) \geq c_n) \sim 1$, and where $w_n(x)$ denotes the Boltzmann weight of the considered model; in the standard REM, this is

$$w_n(x) \equiv \exp\{-\beta H_n^{\mathsf{REM}}(x)\}, \quad x \in \mathcal{V}_n. \tag{1.15}$$

Finally, a common α-stable subordinator emerged as the limit of the clock processes.

As might reasonably be expected, the physical time scale, c_n, on which activated aging occurs in Metropolis dynamics is the same as in the Random Hopping dynamics. What does differ, however, is the choice of K_n. Given a sequence a_n, we now set

$$K_n(t) \equiv \min\left\{ k \geq 1 \,\middle|\, \textstyle\sum_{i=0}^{k-1} \mathbb{1}_{\{J_n(i) \in \mathcal{V}_n \setminus \mathcal{V}_n^\star\}} = \lfloor a_n t \rfloor \right\}, \quad t \geq 0. \tag{1.16}$$

This is the number of steps J_n must take in order to take $\lfloor a_n t \rfloor$ steps outside $\mathcal{V}_n^\star$. Our first theorem states that the resulting rescaled clock process (1.9) converges to the same limiting subordinator as in the Random Hopping dynamics, for the very same sequences a_n and c_n, and in the same β range. For $0 \le \varepsilon \le 1$ and $0 < \beta < \infty$, set

$$\beta_c(\varepsilon) = \sqrt{\varepsilon 2 \log 2}, \tag{1.17}$$

$$\alpha(\varepsilon) = \beta_c(\varepsilon)/\beta. \tag{1.18}$$

Throughout this paper the initial distribution is the uniform distribution on $\mathcal{V}_n \setminus \mathcal{V}_n^\star$.

Theorem 1.1. *Assume that $c_\star > 3$. Given $0 < \varepsilon < 1$ let a_n and c_n be defined through*

$$\lim_{n\to\infty} \frac{\log a_n}{n \log 2} = \varepsilon, \quad a_n \mathbb{P}(w_n(x) \ge c_n) \sim 1. \tag{1.19}$$

Then, for all $0 < \varepsilon < 1$ and all $\beta > \beta_c(\varepsilon)$, $\mathbb{P}$-almost surely,

$$S_n \Rightarrow_{J_1} S_\infty \tag{1.20}$$

where S_∞ is a stable subordinator with zero drift and Lévy measure ν defined through

$$\nu(u, \infty) = u^{-\alpha(\varepsilon)} \alpha(\varepsilon) \Gamma(\alpha(\varepsilon)), \quad u > 0, \tag{1.21}$$

and where $\Rightarrow_{J_1}$ denotes weak convergence in the space $D([0,\infty))$ of càdlàg functions equipped with the Skorokhod J_1-topology.

In the rest of the paper the symbol $\Rightarrow_{J_1}$ (sometimes only $\Rightarrow$) has the same meaning as in Theorem 1.1.

Let us now elucidate the meaning of K_n. There is a clear parallel between the definitions (1.16) and (1.7) of K_n and $\widetilde{S}_n$. Like $\widetilde{S}_n$, K_n is similar to a time, each step of the chain J_n lasting one time unit. Just like $\widetilde{S}_n$ also, it is a function of an underlying 'faster chain', namely, the chain J_n observed only at its visits to $\mathcal{V}_n \setminus \mathcal{V}_n^\star$. Thus K_n can be viewed as the total time spent by the chain J_n along the first $\lfloor a_n t \rfloor$ steps of that fast chain – in other words, as a clock process for J_n. One may probe this parallel further by asking if there exist sequences b_n for which the rescaled process $b_n^{-1} K_n$ converges. As the next theorem shows, the nature of the limit undergoes a transition at the critical value $\beta = 2\beta_c(\varepsilon/2)$.

Theorem 1.2. *Assume that $c_\star > 3$ and, given $0 < \varepsilon < 1$, let a_n be as in Theorem 1.1.*
(i) If $\beta > 2\beta_c(\varepsilon/2)$, let b_n be defined through $\sqrt{n a_n} \mathbb{P}(w_n(x) \ge (n-1) b_n) \sim 1$. Then, for all $0 < \varepsilon < 1$ and all $\beta > 2\beta_c(\varepsilon/2)$, $\mathbb{P}$-almost surely,

$$b_n^{-1} K_n \Rightarrow_{J_1} S_\infty^\dagger, \tag{1.22}$$

where $S_\infty^\dagger$ is a stable subordinator with zero drift and Lévy measure $\nu^\dagger$ defined through

$$\nu^\dagger(u, \infty) = u^{-2\alpha(\varepsilon/2)} 2\alpha(\varepsilon/2) \Gamma(2\alpha(\varepsilon/2)), \quad u > 0. \tag{1.23}$$

(ii) If $0 < \beta < 2\beta_c(\varepsilon/2)$, set $b_n = a_n \exp(n(\beta/2)^2)/(\beta\sqrt{\pi n})$. Then, for all $0 < \varepsilon < 1$ and all $\beta < 2\beta_c(\varepsilon/2)$, $\mathbb{P}$-almost surely,

$$(b_n^{-1} K_n(t), t \ge 0) \overset{P-a.s.}{\underset{n\to\infty}{\longrightarrow}} (t, t \ge 0), \tag{1.24}$$

where convergence holds in the space $C([0,\infty))$ of continuous functions equipped with the topology of the uniform convergence on compact sets.

Remark. A transition similar to that of Theorem 1.2 is present in S_n at the critical value $\beta = \beta_c(\varepsilon)$. Since in the region $\beta < \beta_c(\varepsilon)$ activated aging is interrupted we leave out the explicit statement.

The occurence of stable subordinators as limits of both S_n and $b_n^{-1}K_n$ above the critical lines $\beta = \beta_c(\varepsilon)$ and $\beta = 2\beta_c(\varepsilon/2)$, $0 < \varepsilon < 1$, respectively, can be explained through a single, universal mechanism which is best described as an exploration mechanism of a set of *extreme accessible states* whose effective waiting times are *heavy tailed*. What gives rise to this mechanism, however, is very different depending on whether one considers S_n or $b_n^{-1}K_n$. Let us briefly explain this.

When dealing with S_n, the processes at work are analogous to those already present in the Random Hopping dynamic of the REM: the set of extreme accessible states identifies with the vertices such that $w_n(x) \sim c_n$, and most such vertices belong to the set $\mathcal{I}_n^\star$ of isolated occupied vertices of (1.13), but J_n typically does not revisit the elements of $\mathcal{I}_n^\star$ twice so that the associated effective waiting times typically coincide with the exponential holding times $\lambda_n^{-1}(x)e_{n,i} = w_n(x)e_{n,i}$ (see (1.7)) and these, scaled down by c_n, are asymptotically heavy tailed with parameter $\alpha(\varepsilon)$.

This is in sharp contrast with the mechanisms that govern the behavior of $b_n^{-1}K_n$. Viewing the set $\mathcal{V}_n^\star \cup \mathcal{I}_n^\star$ as the level set of the REM's landscape, and its disjoint components as separated valleys, K_n can be interpreted as the sum of the sojourn times in the valleys of size ≥ 2 that J_n visits along its path. Thus holding times now arise dynamically from metastable trapping times in local valleys. The analysis of these times reveals that the set of extreme accessible states is the set of pairs $(x, y) \in \mathcal{E}_n$ such that $\min(w_n(x), w_n(y)) \sim b_n$, that their effective waiting times have exponential tails of mean value $\min(w_n(x), w_n(y))$, and that, scaled down by b_n, these waiting times are asymptotically heavy tailed with parameter $2\alpha(\varepsilon/2)$.

Below the critical line $\beta = 2\beta_c(\varepsilon/2)$, $0 < \varepsilon < 1$, this picture breaks down. The leading contributions to $b_n^{-1}K_n$ no longer come from extreme events but from *typical* events that consist of visits to valleys whose effective mean waiting times have finite mean values. Note that even here, the jump chain does not resemble the symmetric random walk. In fact, our results show that on the time scales of activated aging, Metropolis dynamics never can be reduced to the Random Hopping dynamics, just as the latter cannot be reduced to Bouchaud's phenomenological trap model. Despite this Bouchaud's trap model does correctly predict the aging behavior of both dynamics:

Theorem 1.3 (Correlation function)**.** *Let $\mathcal{C}_n(t, s)$ be defined in (1.10). Under the hypothesis of Theorem 1.1, for all $\rho \in (0, 1)$, $t > 0$ and $s > 0$, $\mathbb{P}$-almost surely,*

$$\lim_{n\to\infty} \mathcal{P}\big(\mathcal{C}_n(t, s) \geq 1 - \rho\big) = \frac{\sin \alpha\pi}{\pi} \int_0^{t/(t+s)} u^{\alpha(\varepsilon)-1}(1 - u)^{-\alpha(\varepsilon)}\, du. \tag{1.25}$$

Remark. The convergence statement of Theorem 1.2, (i), is of course is a manifestation of the fact that above the critical line the jump chain is itself an aging process. This can be quantified using e.g. the function $\mathcal{C}_n'(t, s) = n^{-1}\big(J_n(\lfloor b_n t\rfloor), J_n(\lfloor b_n(t + s)\rfloor\big)$ for which a statement similar to (1.25) can be proved with $2\alpha(\varepsilon/2)$ subsituted for $\alpha(\varepsilon)$.

Let us highlight the content of the next two sections. What we need to know about the random graph induced by the truncation (1.12) is collected in Section 2. In Section 3 we isolate two processes, called the *front end* and *back end clock processes* (hereafter FECP and BECP), that are central to the proofs of Theorem 1.1 and Theorem 1.2, (i). We show that the processes S_n, respectively K_n, can be written as the sum of FECP, respectively BECP, and remainders. Based on this we decompose the proofs of Theorem 1.1 and Theorem 1.2, (i) into proving on the one hand that FECP and BECP converge, and showing on the other hand that the remainders are asymptotically negligeable. This strategy strongly relies on two abstract theorems (Theorem 8.2 in Section 8 and Theorem 9.1 in Section 9) that give sufficient conditions for FECP and BECP to converge to Lévy subordinators. The proof of Theorem 1.2, (ii) is simpler and relies on classical techniques (standard mean-variance calculations after suitable tructations). Being rather long, we do not present it here to save space. It is given in full detail in an extended version of this work that can be found on arXiv (see http://arxiv.org/pdf/1402.0388.pdf). The organisation of the rest of the paper is detailed at the end of Section 3.

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2. Random Graph properties of the REM's landscape

Given $V \subseteq \mathcal{V}_n$ we denote by $G \equiv G(V)$ the undirected graph which has vertex set V and edge set consisting of pairs of vertices $\{x, y\}$ in V with $\mathrm{dist}(x, y) = 1$. This short section is concerned with the graph properties of the level sets

$$V_n(\rho) = \{x \in \mathcal{V}_n \mid w_n(x) \geq r_n(\rho)\}\,, \tag{2.1}$$

where, given $\rho > 0$, the truncation level $r_n(\rho)$ is the sequence defined through

$$2^{\rho n}\mathbb{P}(w_n(x) \geq r_n(\rho)) = 1. \tag{2.2}$$

This is a convenient reparametrization of (1.12), that is, (1.12) follows from (2.2) by taking

$$\rho = \rho_n^\star \equiv \frac{c_\star \log n}{n \log 2}, \quad r_n(\rho_n^\star) \equiv \exp(\beta\sqrt{n}u_n(c_\star)). \tag{2.3}$$

Viewing the vertices of $\mathcal{V}_n$ as independently occupied with probability $2^{-\rho n}$, questions on $G(V_n(\rho))$ reduce to questions on random subgraphs of the hypercube graph $\mathcal{Q}_n \equiv G(\mathcal{V}_n)$.

2.1. Component structure of $V_n(\rho)$.

The set $V_n(\rho)$ of occupied vertices can be decomposed into components that we classify according to their connectedness and size. We call $C \subseteq V_n(\rho)$ a connected component of size $|C|$ if the subgraph $G(C) \subseteq G(V_n(\rho))$ is connected. By convention, all connected components have size ≥ 2. We call isolated occupied vertices of $V_n(\rho)$ components of size 1. Given $V_n(\rho)$, $\mathcal{V}_n$ can uniquely be decomposed into

$$\mathcal{V}_n = N_n(\rho) \cup I_n(\rho) \cup \left(\cup_{l=1}^L C_{n,l}(\rho)\right), \quad L \equiv L_n(\rho), \tag{2.4}$$

where $N_n(\rho)$ is the set of all non occupied vertices, $I_n(\rho)$ is the set of all isolated occupied ones, and $C_{n,l}(\rho)$, $1 \leq l \leq L$, is a collection of disjoint connected components satisfying

$$G(V_n(\rho)) = \cup_{l=1}^L G(C_{n,l}(\rho)), \quad C_{n,l}(\rho) \cap C_{n,k}(\rho) \ \ \forall l \neq k. \tag{2.5}$$

As ρ decreases, the set $V_n(\rho)$ grows and the graph $G(V_n(\rho))$ potentially acquires new edges. Little is known about such graphs compared to those obtained by selecting edges independently. It is chiefly known [10] that the size of the largest $C_{n,l}(\rho)$ undergoes a transion near the value $\rho \approx \frac{\log n}{n \log 2}$, with a unique "giant" componant of size $\mathcal{O}(n^{-1}2^n)$ emerging slightly below this value. We are interested here in choosing ρ in such a way as to garantee that the size of the largest $C_{n,l}(\rho)$ remains small compared to n. This is done using the next lemma. Define

$$\widetilde{\Omega}_n(m) = \left\{\omega \in \Omega \,\middle|\, \max_{1 \leq l \leq L} |C_{n,l}(\rho)| < m\right\}\,, \quad m = 2, 3, \ldots \tag{2.6}$$

In what follows $\rho \equiv \rho_n > 0$ and $m \equiv m_n > 1$ are, respectively, positive and integer valued sequences. To keep the notation simple we do not make this explicit.

Lemma 2.1. *If $\rho \geq \rho_n^+(m) \equiv \frac{1}{m}\big(1+\frac{(m+2)\log n+\log m!}{n\log 2}\big)$ then $\mathbb{P}\big(\liminf_{n\to\infty}\widetilde{\Omega}_n(m)\big)=1$.*

Proof of Lemma 2.1. Call $(\chi_n(x), x\in\mathcal{V}_n)$, $\chi_n(x)\equiv \mathbb{1}_{\{w_n(x)\geq r_n(\rho)\}}$, the occupancy variables. These are i.i.d. Bernoulli r.v.'s with $\mathbb{P}\left(\chi_n(x)=1\right)=1-\mathbb{P}\left(\chi_n(x)=0\right)=2^{-\rho n}$. Set $\mathbb{P}\big(\widetilde{\Omega}_n^c(m)\big)=1-\mathbb{P}\big(\widetilde{\Omega}_n(m)\big)=\mathbb{P}\big(\exists_{\mathcal{C}_n\subseteq V_n(\rho):|\mathcal{C}_n|=m}\, G(\mathcal{C}_n) \text{ is connected}\big)$. By independence, if $|\mathcal{C}_n|=m$ then $\mathbb{P}\big(G(\mathcal{C}_n) \text{ is connected}\big)=\mathbb{P}\big(\prod_{x\in\mathcal{C}_n}\chi_n(x)=1\big)=(2^{-\rho n})^m$. Furthermore the number of connected components of size m is at most $m!n^m2^n$. To see this choose a vertex $x_0\in\mathcal{V}_n$, and grow a connected component that contains x_0 by adding vertices one by one: since x_0 has n nearest neighborgs there are n ways to add a first vertex, yielding a connected component of size 2; since a connected component of size two has less than $2n$ nearest neighbors there are at most $2n$ ways to add a second vertex, yielding a connected component of size 3, and so on and so forth. Hence, there are at most $n(2n)(3n)\dots(n(m-1))$ ways of growing a component of size m that contains x_0, and since there are 2^n ways of choosing the vertex x_0, the claim follows. Thus, for $\rho\geq\rho_n^+(m)$, $\mathbb{P}\big(\widetilde{\Omega}_n^c(m)\big)\leq m!n^m2^{(1-m\rho)n}\leq m!n^m2^{(1-m\rho_n^+(m))n}\leq n^{-2}$, so that $\sum_{n\geq1}\mathbb{P}\big(\widetilde{\Omega}_n^c(m)\big)<\infty$. The lemma now follows from the first Borel-Cantelli Lemma. $\square$

2.2. **Truncation and related quantities.** Throughout the rest of this section we assume that $c_\star>2$ in (1.12). This will guarantee that a number of needed properties hold true. Stronger conditions on $c_\star>2$ will be needed from Section 6 and beyond. According to (2.4)-(2.5), for $\rho=\rho_n^\star$ as in (2.3), $\mathcal{V}_n$ be decomposed in a unique way into

$$\mathcal{V}_n=N_n^\star\cup I_n^\star\cup\big(\cup_{l=1}^{L^\star}C_{n,l}^\star\big),\quad L^\star\equiv L(\rho_n^\star), \tag{2.7}$$

where $N_n^\star\equiv N_n(\rho_n^\star)$, $I_n^\star\equiv I_n(\rho_n^\star)$, and $C_{n,l}^\star\equiv C_{n,l}(\rho_n^\star)$, $1\leq l\leq L^\star$. By construction $H_n(x)=0$ if and only if $x\in N_n^\star$ (see (1.2) and (1.12)). Furthermore $\mathcal{V}_n^\star$ in (1.13) becomes

$$\mathcal{V}_n^\star=\cup_{l=1}^{L^\star}C_{n,l}^\star. \tag{2.8}$$

Lemma 2.2. *Assume that $c_\star>2$. There exists $\Omega^\star\subset\Omega$ with $\mathbb{P}\left(\Omega^\star\right)=1$ such that on $\Omega^\star$, for all but a finite number of indices n the following holds:*

$$2\leq\left|C_{n,l}^\star\right|\leq\{\rho_n^\star[1-2c_\star^{-1}(1+\mathcal{O}(\log n/n))]\}^{-1},\quad 1\leq l\leq L^\star. \tag{2.9}$$

Furthermore,

$$\begin{aligned}
|I_n^\star| &= 2^nn^{-c_\star}(1-n^{-(c_\star-1)})(1+\mathcal{O}(n^{-2(c_\star-1)})+o(n^{-c_\star})), &(2.10)\\
|V_n(\rho_n^\star)| &= \left|\mathcal{V}_n\setminus N_n^\star\right|=2^nn^{-c_\star}(1+o(n^{-c_\star})), &(2.11)\\
\textstyle\sum_{l=1}^{L^\star}\left|C_{n,l}^\star\right| &= |V_n(\rho_n^\star)\setminus I_n^\star|=2^nn^{-2c_\star+1}(1+\mathcal{O}(n^{-(c_\star-1)})), &(2.12)
\end{aligned}$$

and, setting $\partial_dA\equiv\{y\in\mathcal{V}_n\setminus A:\operatorname{dist}(y,A)=d\}$ where $A\subset\mathcal{V}_n$ and $d=1,2,\dots,$

$$n\left|C_{n,l}^\star\right|(1-\mathcal{O}(\tfrac{1}{\log n}))\leq\left|\partial C_{n,l}^\star\right|\leq n\left|C_{n,l}^\star\right|, \tag{2.13}$$

$$|\partial C_{n,l}^\star\cap\partial x|\geq n(1-\mathcal{O}(\tfrac{1}{\log n}))\ \ \textit{for all}\ \ x\in C_{n,l}^\star, \tag{2.14}$$

$$n\left|C_{n,l}^\star\right|(1-\mathcal{O}(\tfrac{1}{\log n}))\leq\textstyle\sum_{x\in C_{n,l}^\star}\sum_{y\in\partial C_{n,l}^\star:\{x,y\}\in\mathcal{E}_n}1\leq n\left|C_{n,l}^\star\right|. \tag{2.15}$$

Finally, for any integer constant $\kappa_\star>1$ and all $x\in\mathcal{V}_n\setminus\mathcal{V}_n^\star$,

$$\mathbb{P}(\textstyle\sum_{1\leq l\leq L^\star}|\partial x\cap\partial C_{n,l}^\star|\geq\kappa_\star)\leq n^{-\sqrt{\kappa_\star}(2c_\star-3)}+n^{-\sqrt{\kappa_\star+1}(2c_\star-1)+2}, \tag{2.16}$$

$$\mathbb{P}(\textstyle\sum_{1\leq l\leq L^\star}|\partial_2x\cap\partial C_{n,l}^\star|\leq n/\log n)\leq e^{-(2c_\star-3)\sqrt{n\log n}}. \tag{2.17}$$

Proof. The claim of (2.9) follows from Lemma 2.1. Next,

$$|V_n(\rho_n^\star)| = \textstyle\sum_{x\in\mathcal{V}_n}\chi_n(x), \quad |I_n^\star| = \sum_{x\in\mathcal{V}_n}\chi_n(x)\prod_{y\in\mathcal{V}_n:(x,y)\in\mathcal{E}_n}(1-\chi_n(y)), \qquad (2.18)$$

and $\sum_{l=1}^{L^\star}\left|C_{n,l}^\star\right| = \sum_{x\in\mathcal{V}_n}\chi_n(x)[1-\prod_{y\in\mathcal{V}_n:(x,y)\in\mathcal{E}_n}(1-\chi_n(y))]$, where, as in the proof of Lemma 2.1, $\chi_n(x)\equiv \mathbb{1}_{\{w_n(x)\geq r_n(\rho_n^\star)\}}$ are i.i.d. Bernoulli r.v. with $\mathbb{P}\left(\chi_n(x)=1\right)=n^{-c_\star}$. From these expressions (2.11), (2.10), and (2.12) are easily obtained. Turning to (2.15) note that the sum therein can be written as $\sum_{x\in C_{n,l}^\star}(n-d_n(x))$ where $d_n(x)$ denotes the degree of the vertex x in the graph $G(V_n(\rho_n^\star))$. This, the bound $1\leq d_n(x)\leq \left|C_{n,l}^\star\right|$, and (2.9) yield the desired result. To prove the lower bound of (2.13) reason that each vertex x in $C_{n,l}^\star$ has at least $n-d_n(x)$ nearest neighbors vertices in $\partial C_{n,l}^\star$, and that no two vertices in $C_{n,l}^\star$ can have more than one common nearest neighbor vertex in $\partial C_{n,l}^\star$. Hence $\left|\partial C_{n,l}^\star\right|\geq \sum_{x\in C_{n,l}^\star}(n-d_n(x)-(\left|C_{n,l}^\star\right|-1))\geq \sum_{x\in C_{n,l}^\star}(n-2\left|C_{n,l}^\star\right|)$ and the lower bound in (2.13) follows from (2.9). Eq. (2.14) is proved in the same way since $|\partial C_{n,l}^\star\cap\partial x| = n-d(x)$ for $x\in C_{n,l}^\star$. Finally, the upper bound of (2.13) is immediate.

We now prove (2.16) and (2.17). Given $x\in\mathcal{V}_n$ and $i\in\{1,\dots,n\}$, denote by x^i the vertex obtained by flipping the i-th coordinate of x. Similarly, given $i_1,\dots,i_k\in\{1,\dots,n\}$ denote by $x^{i_1\dots i_k}$ the vertex obtained by flipping the coordinates $i_1,\dots,i_k$ successively. Thus, a coordinate that appears a even number of times in the sequence $i_1\dots i_k$ is unchanged, and the distance $\mathrm{dist}(x,x^{i_1\dots i_k})$ is equal to the number of distinct indices. With this notation

$$\textstyle\sum_{1\leq l\leq L^\star}|\partial x\cap\partial C_{n,l}^\star| = \sum_{j_0=1}^n m(j_0)\mathbb{1}_{\{x^{j_0}\in\partial\mathcal{V}_n^\star\}} \qquad (2.19)$$

where $m(j_0)\equiv\sum_{1\leq l\leq L^\star}\mathbb{1}_{\{x^{j_0}\in\partial C_{n,l}^\star\}}$. Since either $\{\forall_{j_0}m(j_0)\leq\kappa_1\}$ or $\{\exists_{j_0}m(j_0)>\kappa_1\}$, writing $\kappa_\star=\kappa_1\kappa_2$, the probability in (2.16) is bounded above by

$$\mathbb{P}(|\partial x\cap\partial\mathcal{V}_n^\star|\geq\kappa_2)+\mathbb{P}(\exists_{j_0}m(j_0)>\kappa_1). \qquad (2.20)$$

Now $|\partial x\cap\partial\mathcal{V}_n^\star| = \sum_{j_0=1}^n \mathbb{1}_{\{\exists 1\leq j_1\neq j_2\neq j_0\leq n:\chi_n(x^{j_0j_1})=1,\chi_n(x^{j_0j_1j_2})=1\}}$ and

$$\mathbb{P}(|\partial x\cap\partial\mathcal{V}_n^\star|\geq\kappa_\star) = \textstyle\sum_{k\geq\kappa_2}\binom{n}{k}q_n^k(1-q_n)^{n-k} \qquad (2.21)$$

where $q_n=\mathbb{P}(\exists 1\leq j_1\neq j_2\neq j_0\leq n:\chi_n(x^{j_0j_1})=1,\chi_n(x^{j_0j_1j_2})=1)$. Using Poincaré inclusion-exclusion formula to evaluate q_n then yields, given that $2c_\star>3$,

$$\mathbb{P}(|\partial x\cap\partial\mathcal{V}_n^\star|\geq\kappa_2) = \tbinom{n}{\kappa_2}\left(\tfrac{(n-1)(n-2)}{2n^{2c_\star}}\right)^{\kappa_2}(1+o(1)). \qquad (2.22)$$

Next, the second probability in (2.20) is bounded above by

$$\mathbb{P}\left(\exists_{j_0,(j_1,j_1'),\dots,(j_{\kappa_1+1},j_{\kappa_1+1}')}\forall_{1\leq i\leq\kappa_1+1}\chi_n(x^{j_0j_i})=1,\chi_n(x^{j_0j_ij_i'})=1\right) \qquad (2.23)$$

where the (j_i,j_i')'s are are such that the vertices $\{x^{j_0j_i},x^{j_0j_ij_i'},1\leq i\leq\kappa_1+1\}$ are all distinct and distinct from x^{j_0}. Thus, by independence,

$$\mathbb{P}(\exists_{j_0}m(j_0)>\kappa_1)\leq n^2\tbinom{n}{\kappa_1+1}n^{-2(\kappa_1+1)c_\star} \qquad (2.24)$$

Plugging (2.22) and (2.24) in (2.20) and taking $\kappa_1=\kappa_2=\sqrt{\kappa_\star}$ yields (2.16). Eq. (2.17) is proved in the same way. □

We conclude this section with two elementary lemmata that are repeatedly needed. The first expresses the function $r_n(\rho)$ defined through (2.2).

Lemma 2.3. *For all $\rho>0$, possibly depending on n, such that $\rho n\uparrow\infty$ as $n\uparrow\infty$,*

$$r_n(\rho)=\exp\left\{n\beta\beta_c(\rho)-(\beta/2\beta_c(\rho))\left[\log(\beta_c^2(\rho)n/2)+\log 4\pi\right]+o(\beta/\beta_c(\rho))\right\}. \qquad (2.25)$$

In particular, for $\rho_n^\star$ as in (2.3) and $c_\star > 2$,

$$r_n\big(\rho_n^\star\big) = \exp\Big\{\beta\Big(\sqrt{2c_\star n\log n} - \sqrt{\tfrac{n}{\log n}}\big(\tfrac{\log\log n}{2\sqrt{2c_\star}} + \mathcal{O}(1)\big)\Big)\Big\}. \tag{2.26}$$

Proof. Denote by Φ and ϕ the standard Gaussian distribution function and density, respectively. Setting $b_n = 2^{\rho n}$ and $\overline{B}_n = \log r_n(\rho)/\beta\sqrt{n}$, (2.2) becomes $b_n\big(1-\Phi(\overline{B}_n)\big) = 1$. It is shown in [25] (see paragraph below (2.20)) that $(\overline{B}_n - B_n)B_n = o(1)$ where B_n is defined through $b_n\frac{\phi(B_n)}{B_n} = 1$. Eq. (2.25) then readily follows from the well known fact that (see [18], p. 374) $B_n = (2\log b_n)^{\frac12} - \frac12(\log\log b_n + \log 4\pi)/(2\log b_n)^{\frac12} + \mathcal{O}(1/\log b_n)$. □

Lemma 2.4. *There exists a subset $\Omega_0 \subseteq \Omega$ with $\mathbb{P}\big(\Omega_0\big) = 1$ such that on Ω_0, for all but a finite number of indices n the following holds: for all $1 \le l \le L^\star$*

$$e^{-\beta\min\left\{\max(H_n(y),H_n(x))\,|\,\{x,y\}\in G(C^\star_{n,l})\right\}} \le e^{\beta n\sqrt{\log 2}(1+2\log n/n\log 2)}, \tag{2.27}$$

$$e^{-\beta\min\left\{H_n(x)\,|\,x\in C^\star_{n,l}\right\}} \le e^{\beta n\sqrt{2\log 2}(1+2\log n/n)}. \tag{2.28}$$

Proof. Set $\rho(1) \equiv 1 + 2\log n/n\log 2$, $\Omega_n(1) \equiv \{\omega \in \Omega \mid \max_{x\in\mathcal{V}_n} w_n(x) \le r_n(\rho(1))\}$, and $\Omega_\infty(1) \equiv \liminf_{n\to\infty}\Omega_n(1)$. Further set $\rho(2) \equiv \frac12(1 + 3\log n/n\log 2)$, $\Omega_n(2) \equiv \{\omega \in \Omega \mid \max_{(x,y)\in\mathcal{E}_n}\min(w_n(x), w_n(y)) \le r_n(\rho(2))\}$, and $\Omega_\infty(2) \equiv \liminf_{n\to\infty}\Omega_n(2)$. By independence and (2.2), $\mathbb{P}(\Omega_n^c(1)) = 2^n 2^{-n} n^{-2} = n^{-2}$ which is summable, hence $\mathbb{P}(\Omega_\infty(1)) = 1$. Next, $\mathbb{P}(\Omega_n^c(2)) \le n2^{n-1}2^{-n}(r_n(\rho(2)))^2 \le n^{-2}$ which is also summable, and so $\mathbb{P}(\Omega_\infty(2)) = 1$. Taking $\Omega_0 \equiv \Omega_\infty(1)\cap\Omega_\infty(2)$ and using (2.25) to bound $r_n(\rho(1))$ and $r_n(\rho(2))$ yields the claim of the lemma. □

3. Front end and back end clock processes, and proofs of the theorems of Section 1.

In this section we formally define the *front end* and *back end* clock processes, and show how they relate to the clock processes S_n and K_n. These relations are then used to decompose the proofs of Theorem 1.1 and Theorem 1.2 into five main steps. Let $C^\star_{n,l}$, $1 \le l \le L^\star$, be the collection of connected components defined through (2.7) and set

$$\mathcal{V}_n^\circ \equiv \mathcal{V}_n \setminus \left(\cup_{1\le l\le L^\star} C^\star_{n,l}\right). \tag{3.1}$$

3.1. Front end clock process.

We call *front end clock process* the process defined through

$$\widetilde{S}_n^\circ(k^\circ) = \sum_{i=0}^{k^\circ-1}\lambda_n^{-1}(J_n^\circ(i))e^\circ_{n,i}, \quad k^\circ \in \mathbb{N}, \tag{3.2}$$

where $(e^\circ_{n,i}, n \in \mathbb{N}, i \in \mathbb{N})$ are independent mean one exponential random variables and where, introducing the times of consecutive visits of J_n to $\mathcal{V}_n^\circ$,

$$T^\circ_{n,0} = \inf\{i \ge 0 \mid J_n(i) \in \mathcal{V}_n^\circ\}, \tag{3.3}$$

$$T^\circ_{n,j+1} = \inf\{i > T^\circ_{n,j} \mid J_n(i) \in \mathcal{V}_n^\circ\},\ j = 0, 1, 2, \dots, \tag{3.4}$$

$(J_n^\circ(i), i \in \mathbb{N})$ is the reversible Markov chain on $\mathcal{V}_n^\circ$ obtained by setting $J_n^\circ(i) \equiv J_n(T^\circ_{n,i})$. Note that J_n° has transition matrix elements

$$p_n^\circ(x, y) = P_x\left(J_n(T^\circ_{n,1}) = y\right),\ x, y \in \mathcal{V}_n^\circ, \tag{3.5}$$

and invariant measure

$$\pi_n^\circ(x) = \frac{\pi_n(x)}{\sum_{x'\in\mathcal{V}_n^\circ}\pi_n(x')}, \quad x \in \mathcal{V}_n^\circ, \tag{3.6}$$

where π_n denotes the invariant measure of J_n (see (6.8) for its expression). We call J_n° the front chain and denote by $(\Omega^{J^\circ}, \mathcal{F}^{J^\circ}, P^\circ)$ its probability space. The associated graph, $G^\circ(\mathcal{V}_n^\circ)$, is described in (6.11).

3.2. **Back end clock process.** The description of this process involves three time sequences. The first two are the intertwined sequences of consecutive hitting times of $\mathcal{V}_n \setminus \mathcal{V}_n^\circ$ and their ensuing exit times. Namely, set

$$\overline{T}_{n,0} = 0, \quad \overline{T}'_{n,0} = \begin{cases} \inf\{i > 0 \mid J_n(i) \notin \mathcal{V}_n^\circ\}, & \text{if } J_n(0) \in \mathcal{V}_n^\circ, \\ 0, & \text{if } J_n(0) \notin \mathcal{V}_n^\circ, \end{cases} \tag{3.7}$$

and, for $j = 0, 1, 2, \dots,$

$$\overline{T}_{n,j+1} = \inf\{i > \overline{T}'_{n,j} \mid J_n(i) \in \mathcal{V}_n^\circ\}, \tag{3.8}$$

$$\overline{T}'_{n,j+1} = \inf\{i > \overline{T}_{n,j+1} \mid J_n(i) \notin \mathcal{V}_n^\circ\}. \tag{3.9}$$

Clearly, $0 = \overline{T}_{n,0} \le \overline{T}'_{n,0} < \overline{T}_{n,1} \le \overline{T}'_{n,1} < \cdots < \overline{T}_{n,j} \le \overline{T}'_{n,j} < \dots$. Clearly also, to each j there corresponds an i such that $T^\circ_{n,i-1} < \overline{T}'_{n,j} = T^\circ_{n,i-1} + 1 < T^\circ_{n,i}$. Merging $\left(T^\circ_{n,i}\right)_{i\ge 0}$ and $\left(\overline{T}'_{n,j}\right)_{j\ge 0}$ into a single sequence, $\left(T^\dagger_{n,j}\right)_{j\ge 0}$, and arranging its elements in increasing order of magnitude,

$$0 \le T^\dagger_{n,0} < T^\dagger_{n,1} < \cdots < T^\dagger_{n,j} < \dots. \tag{3.10}$$

we define the *back end clock process* through

$$\widetilde{S}_n^\dagger(k^\dagger) = \sum_{i=0}^{k^\dagger - 1} \Lambda_n^\dagger(i), \quad k^\dagger \in \mathbb{N}, \tag{3.11}$$

where, denoting by $(J_n^\dagger(i), i \in \mathbb{N})$ the chain on $\mathcal{V}_n$ obtained by setting $J_n^\dagger(i) \equiv J_n(T^\dagger_{n,i})$,

$$\Lambda_n^\dagger(i) = \begin{cases} \overline{T}_{n,j+1} - \overline{T}'_{n,j}, & \text{if } J_n^\dagger(i) \notin \mathcal{V}_n^\circ \text{ and } \sum_{k=0}^{i} \mathbb{1}_{\{J_n^\dagger(k) \notin \mathcal{V}_n^\circ\}} = j, \\ 0, & \text{if } J_n^\dagger(i) \in \mathcal{V}_n^\circ. \end{cases} \tag{3.12}$$

Clearly, $J_n^\dagger$ is Markovian with one-step transitions probabilities, $p_n^\dagger(x, y)$, as follows: when it is at $x \in \mathcal{V}_n^\circ$, $J_n^\dagger$ chooses its next step according to the transition probabilities of J_n,

$$p_n^\dagger(x, y) = p_n(x, y), \quad x \in \mathcal{V}_n^\circ, y \in \mathcal{V}_n, \tag{3.13}$$

and when it enters $\cup_{1\le l\le L^\star} C^\star_{n,l}$, say at a vertex of $C^\star_{n,l}$, it exits in just one step through one of the boundary points $\partial C^\star_{n,l}$; that is, for all $x \in C^\star_{n,l}$, $y \in \partial C^\star_{n,l}$, and $1 \le l \le L^\star$,

$$p_n^\dagger(x, y) = P_x(J(T^\star_{n,l}) = y), \tag{3.14}$$

where $T^\star_{n,l} = \inf\{i > 0 \mid J_n(i) \in \partial C^\star_{n,l}\}$. Clearly also, the increments $\Lambda_n^\dagger(i)$ of the clock at the times of the visits of $J_n^\dagger(i)$ to $\cup_{1\le l\le L^\star} C^\star_{n,l}$ are the sojourn times of J_n in the sets $C^\star_{n,l}$ being visited. In other words, $\Lambda_n^\dagger(i)$ is equal in distribution to some $T^\star_{n,l}$.

Summarizing our definitions, FECP (3.2) records the total time spent by the proces X_n in $\mathcal{V}_n^\circ$ along the first k° steps of J_n° whereas BECP (3.11) records the total time spent by the chain J_n in $\cup_{1\le l\le L^\star} C^\star_{n,l}$ along the first $k^\dagger$ steps of $J_n^\dagger$. The chains $J_n^\dagger$ and J_n° differ in that $J_n^\dagger$ does visit the sets $C^\star_{n,l}$, and steps out of these sets right after stepping in, while J_n° straddles over the $C^\star_{n,l}$'s, never entering them. Technically, this makes the two chains very different objects. In particular, J_n° is reversible but $J_n^\dagger$ isn't.

3.3. **Rewriting the clock process.** Our aim is to express the processes K_n and S_n defined in (1.16) and (1.9), respectively, using FECP and BECP. We first deal with K_n. For a_n as in (1.16) let $k_n^\dagger(t)$ be defined through

$$k_n^\dagger(t) = \min\left\{k \geq 1 \,\middle|\, \textstyle\sum_{i=0}^{k-1} \mathbb{1}_{\{J_n^\dagger(i)\in\mathcal{V}_n^\circ\}} = \lfloor a_n t\rfloor\right\}, \quad t \geq 0, \tag{3.15}$$

and, taking $k^\dagger = k_n^\dagger(t)$ in (3.11), set

$$S_n^\dagger(t) = b_n^{-1}\widetilde{S}_n^\dagger(k_n^\dagger(t)), \quad t \geq 0, \tag{3.16}$$

where b_n is a sequence to be chosen (and ultimately chosen as in Theorem 1.2). $K_n(t)$ can then be writen as

$$K_n(t) = \lfloor a_n t\rfloor + b_n S_n^\dagger(t), \quad t \geq 0. \tag{3.17}$$

To see this write $K_n(t) = \sum_{i=0}^{K_n(t)-1} \mathbb{1}_{\{J_n(i)\in\mathcal{V}_n^\circ\}} + \sum_{i=0}^{K_n(t)-1} \mathbb{1}_{\{J_n(i)\notin\mathcal{V}_n^\circ\}}$ and note that

$$\textstyle\sum_{i=0}^{K_n(t)-1} \mathbb{1}_{\{J_n(i)\notin\mathcal{V}_n^\circ\}} = \sum_{i=0}^{k_n^\dagger(t)-1} \Lambda_n^\dagger(i) = b_n S_n^\dagger(t), \tag{3.18}$$

$$\textstyle\sum_{i=0}^{K_n(t)-1} \mathbb{1}_{\{J_n(i)\in\mathcal{V}_n^\circ\}} = \sum_{i=0}^{k_n^\dagger(t)-1} \mathbb{1}_{\{J_n^\dagger(i)\in\mathcal{V}_n^\circ\}} = \lfloor a_n t\rfloor \equiv k_n^\circ(t), \tag{3.19}$$

where we introduced the notation $k_n^\circ(t)$ for later convenience. In words, when J_n takes $K_n(t)$ steps, $J_n^\dagger$ takes $k_n^\dagger(t)$ steps, of which $k_n^\circ(t)$ are visits of $J_n^\dagger$ to $\mathcal{V}_n^\circ$.

To deal with the clock process S_n we likewise split the sum in (1.9) in two terms according to whether $J_n(i) \in \mathcal{V}_n^\circ$ or $J_n(i) \notin \mathcal{V}_n^\circ$. From the above definitions and those of $J_n^\dagger$ and J_n° we have that on the one hand, writing $\stackrel{d}{=}$ for equality in distribution,

$$\begin{aligned}
\textstyle\sum_{i=0}^{K_n(t)-1} \lambda_n^{-1}(J_n(i))e_{n,i}\mathbb{1}_{\{J_n(i)\in\mathcal{V}_n^\circ\}} &\stackrel{d}{=} \textstyle\sum_{j=0}^{k_n^\dagger(t)-1} \lambda_n^{-1}(J_n^\dagger(j))e_{n,j}^\dagger\mathbb{1}_{\{J_n^\dagger(i)\in\mathcal{V}_n^\circ\}} &(3.20)\\
&\stackrel{d}{=} \textstyle\sum_{j=0}^{k_n^\circ(t)-1} \lambda_n^{-1}(J_n^\circ(j))e_{n,j}^\circ\mathbb{1}_{\{J_n^\circ(i)\in\mathcal{V}_n^\circ\}} &(3.21)\\
&= \widetilde{S}_n^\circ(\lfloor a_n t\rfloor), &(3.22)
\end{aligned}$$

where $(e_{n,j}^\dagger)$ and $(e_{n,j}^\circ)$ are families of independent mean one exponential random variables, and $\widetilde{S}_n^\circ$ is the front end clock process (3.2). On the other hand,

$$\begin{aligned}
&\textstyle\sum_{i=0}^{K_n(t)-1} \lambda_n^{-1}(J_n(i))e_{n,i}\mathbb{1}_{\{J_n(i)\notin\mathcal{V}_n^\circ\}} &(3.23)\\
\stackrel{d}{=}\ &\textstyle\sum_{j=0}^{k_n^\dagger(t)-1} \left(\sum_{i=0}^{\Lambda_n^\dagger(j)-1} \lambda_n^{-1}(J_n(\overline{T}'_{n,j}+i))e_{n,j,i}\right)\mathbb{1}_{\{J_n^\dagger(j)\notin\mathcal{V}_n^\circ\}} &(3.24)\\
\equiv\ &\textstyle\sum_{j=0}^{k_n^\dagger(t)-1} \widehat{\Lambda}_n^\dagger(j) &(3.25)
\end{aligned}$$

where the last line defines $\widehat{\Lambda}_n^\dagger(j)$, and where $(e_{n,j,i})$ are independent mean one exponential random variables. If we now set, for $t \geq 0$,

$$\begin{aligned}
S_n^\circ(t) &\equiv c_n^{-1}\widetilde{S}_n^\circ(\lfloor a_n t\rfloor), &(3.26)\\
\widehat{S}_n(t) &\equiv c_n^{-1}\textstyle\sum_{j=0}^{k_n^\dagger(t)-1} \widehat{\Lambda}_n^\dagger(j), &(3.27)
\end{aligned}$$

the rescaled clock process (1.9) can be rewritten as

$$S_n(t) \stackrel{d}{=} S_n^\circ(t) + \widehat{S}_n(t). \tag{3.28}$$

Here the rescaled front end clock process, $S_n^\circ(t)$, records the time spent *by the process* X_n during its visits to the set $\mathcal{V}_n^\circ$, while the remainder term, $\widehat{S}_n(t)$, records the time spent in its complement. The back end clock process $b_n S_n^\dagger(t)$ is the time needed to actually be able to observe a transition *of the chain* J_n from one vertex of $\mathcal{V}_n^\circ$ to the next.

3.4. **Proof of Therorem 1.1 and Therorem 1.2.** The proofs of Theorem 1.1 and Theorem 1.2 rely on four theorems stated below. Each of them controls one of the processes $k_n^\dagger(t)$, $S_n^\circ(t)$, $\widehat{S}_n(t)$, and $S_n^\dagger(t)$ above, respectively below, the critical line $\beta = 2\beta_c(\varepsilon/2)$, $0 < \varepsilon < 1$. As in Section 1.2 the initial distribution of J_n is the uniform distribution on $\mathcal{V}_n^\circ$. By (6.6), this is nothing but the invariant measure, π_n°, of J_n°. Thus J_n° and $J_n^\dagger$ also start in π_n°.

The first theorem shows that $k_n^\dagger(t)$ behaves like $k_n^\circ(t) = \lfloor a_n t \rfloor$ for large n.

Theorem 3.1. *Assume that $c_\star > 2$. For all $0 < t < \infty$, any constant $c_\circ > 0$, and any sequence $a_n > 0$ we have that on $\Omega^\star$, for all but a finite number of indices n,*

$$P_{\pi_n^\circ}\left(1 \leq k_n^\dagger(t)/k_n^\circ(t) \leq 1 + n^{-c_\circ}\right) \geq 1 - n^{-2(c_\star-1)+c_\circ}(1 + \mathcal{O}(n^{-(c_\star-1)})). \tag{3.29}$$

The next two theorems are the building blocks of the proof of Theorem 1.1. The first establishes convergence of the front end clock process, S_n°. The second implies, in particular, that the contribution of $\widehat{S}_n$ to (3.28) vanishes as n diverges.

Theorem 3.2 (Front end clock process)**.** *Assume that $c_\star > 3$. Let the sequences a_n and c_n be as in Theorem 1.1. Then, for all $0 < \varepsilon < 1$ and $\beta > \beta_c(\varepsilon)$, $\mathbb{P}$-almost surely,*

$$S_n^\circ \Rightarrow_{J_1} S_\infty^\circ, \tag{3.30}$$

where S_∞° is a subordinator with zero drift and Lévy measure $\nu^\circ = \nu$ defined in (1.21).

Theorem 3.3 (Remainder)**.** *Assume that $c_\star > 2$. Let the sequences a_n and c_n be as in Theorem 1.1. Then, for all $0 < \varepsilon < 1$ and $\beta > \beta_c(\varepsilon)$, $\mathbb{P}$-almost surely,*

$$\limsup_{n\to\infty} \mathcal{P}_{\pi_n^\circ}\left(\rho_\infty\big(S_n(\cdot), S_n^\circ(\cdot)\big) > n^{1-c_\star/2}\right) = 0, \tag{3.31}$$

where ρ_∞ is Skorohod metric on $D([0,\infty))$.

We now turn to the back end clock process. The next result parallels Theorem 3.2.

Theorem 3.4 (Back end clock process above the critical line)**.** *Assume that $c_\star > 3$. Let the sequence a_n and b_n be as in Theorem 1.1 and Theorem 1.2, (i), respectively. Then, for all $0 < \varepsilon < 1$ and $\beta > 2\beta_c(\varepsilon/2)$, $\mathbb{P}$-almost surely,*

$$S_n^\dagger \Rightarrow_{J_1} S_\infty^\dagger, \tag{3.32}$$

where $S_\infty^\dagger$ is a stable subordinator with zero drift and Lévy measure $\nu^\dagger$ defined in (1.23).

Assuming these theorems we may prove Theorem 1.1 and Theorem 1.2. The proof of Theorem 1.3 that also uses from Section 6 and Section 7 is postponed to Section 7.

Proof of Theorem 1.1. In view of (3.28) Theorem 1.1 is an immediate consequence of Theorem 3.2 and Theorem 3.3 □

Proof of Theorem 1.2. Recall the expression (3.17) of K_n and notice that $a_n/b_n \downarrow 0$ under the assumptions on a_n and b_n of Theorem 3.4 (use (2.25) to check this). Thus the first assertion of Theorem 1.2 is a an immediate consequence of Theorem 3.4. See the extended version of this paper on arXiv (http://arxiv.org/pdf/1402.0388.pdf) for the proof of the second assertion. See in particular Theorem 3.5 therein and its proof. □

The rest of this paper is organized as follows. In Section 4 we focus on the increments of the process $\widehat{S}_n$ and prove an upper bound on their tail distribution. A similar analysis is carried out in Section 5 for the increments of the back end clock process $\widetilde{S}_n^\dagger$; an explicit expression is also obtained for the distribution of the sojourn times of J_n in sets $C_{n,l}^\star$ of size 2. The properties of J_n° (invariant measure, mixing time through spectral gap, mean local times) are studied in Section 6, where it is shown that J_n° has several of the attributes of the symmetric random walk. Using these preparations, the proofs of Theorem 3.1 and Theorem 3.3, as well as that of Theorem 1.3 are carried out in Section 7. Those of Theorem 3.2 and Theorem 3.4 are carried out in Section 8 and Section 9, respectively.

4. Distribution of the increments of the process $\widehat{S}_n$.

In this section we focus on the increments of the process $\widehat{S}_n$, that is to say, on the quantities defined through (3.24)-(3.25) by

$$\widehat{\Lambda}_n^\dagger(j) \equiv \sum_{i=0}^{\Lambda_n^\dagger(j)-1} \lambda_n^{-1}(J_n(\overline{T}'_{n,j}+i))e_{n,j,i} \tag{4.1}$$

if $J_n^\dagger(j) \in \cup_{1\le l\le L^\star} C_{n,l}^\star$, and $\widehat{\Lambda}_n^\dagger(j) = 0$ otherwise. These are the sojourn times of the process X_n in the sets $C_{n,l}^\star$ (we may think of them as "effective holding times" in those sets). As expected, these times have exponential tails. For $1 \le l \le L^\star$, set

$$\bar{\varrho}_{n,l}(0) = e^{-\beta \min\left\{H_n(x)\,|\,x\in C_{n,l}^\star\right\}}. \tag{4.2}$$

Proposition 4.1. *On $\Omega^\star$ (for $\Omega^\star$ as in Lemma 2.2), for all but a finite number of indices n, the following holds for all $1 \le l \le L^\star$: for all $t \ge 0$ and all x in $C_{n,l}^\star$*

$$\mathcal{P}\left(\widehat{\Lambda}_n^\dagger(j) > t \mid J_n^\dagger(j) = x\right) \le e^{-t(1-|C_{n,l}^\star|/n)/\bar{\varrho}_{n,l}(0)}. \tag{4.3}$$

The next corollary is a key ingredient of the proof of Theorem 3.3.

Corollary 4.2. *Assume that $a_n \le 2^n$. On $\Omega^\star$, for all but a finite number of indices n,*

$$\mathcal{P}_{\pi_n^\circ}\left(\exists_{0\le j\le k_n^\dagger(t)-1}\exists_{1\le l\le L^\star}\widehat{\Lambda}_n^\dagger(j)\mathbb{1}_{\{J_n^\dagger(j)\in C_{n,l}^\star\}} > 2n\bar{\varrho}_{n,l}(0)\right) \le te^{-n} + n^{-2(c_\star-1)+c_\circ} \tag{4.4}$$

where $c_\circ > 0$ is a constant that can be chosen arbitrarily small.

Proof of Proposition 4.1. Let $C_{n,l}^\star$, $1 \le l \le L^\star$, be the collection of connected components defined through (2.7). To each component $C_{n,l}^\star$ we associate an absorbing Markov process $X_{n,l}^\star$ with state space $C_{n,l}^\star \cup \Delta$, where the absorbing point, Δ, represents the boundary $\partial C_{n,l}^\star$; its infinitesimal generator $\overline{\mathcal{L}}_{n,l}^\star = \left(\bar{\lambda}_{n,l}^\star(x,y)\right)$ has entries $\bar{\lambda}_{n,l}^\star : \{C_{n,l}^\star \cup \Delta\} \times \{C_{n,l}^\star \cup \Delta\} \to \mathbb{R}$, given by

$$\bar{\lambda}_{n,l}^\star(x,y) = \begin{cases} \lambda_n(x,y) & \text{if} \quad (x,y) \in G(C_{n,l}^\star), \\ \sum_{y'\notin C_{n,l}^\star} \lambda_n(x,y') & \text{if} \quad x \in C_{n,l}^\star, y = \Delta, \\ -\sum_{y'\in\mathcal{V}_n} \lambda_n(x,y') & \text{if} \quad x = y \in C_{n,l}^\star, \\ 0 & \text{else.} \end{cases} \tag{4.5}$$

Thus $X_{n,l}^\star$ can be viewed as the restriction of X_n to $C_{n,l}^\star$, killed on the boundary $\partial C_{n,l}^\star$. We also call $\mathcal{L}_{n,l}^\star = \left(\lambda_{n,l}^\star(x,y)\right)$ the sub-Markovian restriction of $\overline{\mathcal{L}}_{n,l}^\star$ to $C_{n,l}^\star$, namely $\lambda_{n,l}^\star : C_{n,l}^\star \times C_{n,l}^\star \to \mathbb{R}$,

$$\lambda_{n,l}^\star(x,y) = \begin{cases} \lambda_n(x,y) & \text{if} \quad (x,y) \in G(C_{n,l}^\star) \\ -\sum_{y'\in\mathcal{V}_n} \lambda_n(x,y') & \text{if} \quad x = y \in C_{n,l}^\star. \end{cases} \tag{4.6}$$

With this notation $\widehat{\Lambda}_n^\dagger(j)$ in (4.1) is nothing but the absorption time

$$\Lambda_{n,l}^\star \equiv \inf\{t > 0 \mid X_{n,l}^\star(t) = \Delta\} \tag{4.7}$$

of the process $X^\star_{n,l}$ started in $X^\star_{n,l}(0) = J^\dagger_n(j)$. Furthermore, for all $x \in C^\star_{n,l}$ and $t > 0$,

$$\mathcal{P}_x\left(\Lambda^\star_{n,l} > t\right) = \sum_{y\in C^\star_{n,l}}(\delta_x, e^{t\mathcal{L}^\star_{n,l}}\delta_y) \tag{4.8}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$, $N \equiv |C^\star_{n,l}|$, and δ_x is the vector with components $\delta_x(x') = 1$ if $x' = x$ and zero otherwise. Denoting by I_N the identity and by $R_{n,l} = (r_{n,l}(x, y))$ the nonnegative matrix $R_{n,l} \equiv \mathcal{L}^\star_{n,l} + I_N$, (4.8) can be written as

$$\mathcal{P}_x\left(\Lambda^\star_{n,l} > t\right) = \sum_{k=0}^\infty \tfrac{t^k}{k!}e^{-t}\sum_{y\in C^\star_{n,l}}(\delta_x, R^k_{n,l}\delta_y). \tag{4.9}$$

where, in explicit form, for each $k \geq 1$,

$$\sum_{y\in C^\star_{n,l}}(\delta_x, R^k_{n,l}\delta_y) = \sum_{x_1\in C^\star_{n,l}} r_{n,l}(x, x_1)\cdots\sum_{x_k\in C^\star_{n,l}} r_{n,l}(x_{k-1}, x_k). \tag{4.10}$$

Consider the last sum in (4.10) and observe that by (4.6) and (1.4), for all $x_{k-1} \in C^\star_{n,l}$,

$$\begin{aligned}\sum_{x_k\in C^\star_{n,l}} r_{n,l}(x_{k-1}, x_k) &= 1 - \sum_{x_k\notin C^\star_{n,l}} \lambda_n(x_{k-1}, x_k) &(4.11)\\ &= 1 - (1 - |C^\star_{n,l}|n^{-1})e^{\beta H_n(x_{k-1})} &(4.12)\\ &\leq 1 - (1 - |C^\star_{n,l}|n^{-1})/\bar\varrho_{n,l}(0) &(4.13)\end{aligned}$$

where $\bar\varrho_{n,l}(0)$ is as in (4.2). Inserting (4.13) in (4.10) and iterating leads to

$$\sum_{y\in C^\star_{n,l}}(\delta_x, R^k_{n,l}\delta_y) \leq [1 - (1 - |C^\star_{n,l}|n^{-1})/\bar\varrho_{n,l}(0)]^k \tag{4.14}$$

which, in turn, inserted in (4.9) yields

$$\mathcal{P}_x\left(\Lambda^\star_{n,l} > t\right) \leq e^{-t}e^{t\left[1-\left(1-|C^\star_{n,l}|n^{-1}\right)/\bar\varrho_{n,l}(0)\right]} = e^{-t(1-|C^\star_{n,l}|n^{-1})/\bar\varrho_{n,l}(0)} \tag{4.15}$$

and proves (4.3). □

Proof of Corollary 4.2. By Proposition 4.1 with $t = 2n\bar\varrho_{n,l}(0)$, on $\Omega^\star$, for all but a finite number of indices n

$$\mathcal{P}\left(\widehat\Lambda^\dagger_n(j) > 2n\bar\varrho_{n,l}(0) \mid J^\dagger_n(j) = x\right) \leq e^{-2n(1-o(1))} \tag{4.16}$$

for all $1 \leq l \leq L^\star$ and all $x \in C^\star_{n,l}$. Let $\mathcal{A}$ be the event in the left hand side of (4.4). By Theorem 3.1, $\mathcal{P}_{\pi^\circ_n}(\mathcal{A}) \leq \mathcal{P}_{\pi^\circ_n}(\mathcal{A}, \{k^\dagger_n(t) \leq \lfloor a_n t\rfloor(1 + n^{-c_\circ})\}) + 2n^{-2(c_\star-1)+c_\circ}$, and by (4.16), on $\Omega^\star$, $\mathcal{P}_{\pi^\circ_n}(\mathcal{A}, \{k^\dagger_n(t) \leq \lfloor a_n t\rfloor(1 + n^{-c_\circ})\}) \leq 2\lfloor a_n t\rfloor(1 + n^{-c_\circ})e^{-2n(1-o(1))}$ for all but a finite number of indices n. Since $a_n < 2^n$, (4.4) follows. □

5. DISTRIBUTION OF THE INCREMENTS OF THE BACK END CLOCK PROCESS $\widetilde S^\dagger_n$.

This section parallels Section 4, focusing this time on the increments, $\Lambda^\dagger_n$, defined in (3.12), of the process $\widetilde S^\dagger_n$. Just as the $\widehat\Lambda^\dagger_n$'s are the sojourn times of the process X_n in the sets $C^\star_{n,l}$, the $\Lambda^\dagger_n$'s are the sojourn times of the chain J_n in those sets. For $1 \leq l \leq L^\star$, set

$$\varrho_{n,l}(0) = e^{-\beta\min\left\{\max(H_n(y),H_n(x))\,|\,\{x,y\}\in G(C^\star_{n,l})\right\}}. \tag{5.1}$$

Proposition 5.1. *(i) For each $1 \leq l \leq L^\star$ such that $|C^\star_{n,l}| = 2$ we have, for all $i > 0$ and all x in $C^\star_{n,l}$,*

$$\mathcal{P}\left(\Lambda^\dagger_n(j) > i \mid J^\dagger_n(j) = x\right) = \left(1 - \tfrac{1}{1+\varrho_{n,l}(0)/(n-1)}\right)^i. \tag{5.2}$$

(ii) Furthermore, on $\Omega^\star$ (for $\Omega^\star$ as in Lemma 2.2), for all but a finite number of indices n, the following holds for all $1 \leq l \leq L^\star$: for all $i \geq 0$ and all x in $C^\star_{n,l}$

$$\mathcal{P}\left(\Lambda^\dagger_n(j) > i \mid J^\dagger_n(j) = x\right) \leq e^{-i(n/\varrho_{n,l}(0)|C^\star_{n,l}|)(1-o(1))}. \tag{5.3}$$

Proof of Proposition 5.1. Let $C^\star_{n,l}$, $1 \leq l \leq L^\star$, be the collection of connected components defined through (2.7). To each component $C^\star_{n,l}$ we associate an absorbing Markov chain $J^\star_{n,l}$ with state space $C^\star_{n,l} \cup \Delta$, where the absorbing point Δ represents the boundary $\partial C^\star_{n,l}$; its transition matrix $P^\star_{n,l} = \left(p^\star_{n,l}(x,y)\right)$ has entries $p^\star_{n,l} : \{C^\star_{n,l} \cup \Delta\} \times \{C^\star_{n,l} \cup \Delta\} \to [0,1]$,

$$p^\star_{n,l}(x,y) = \begin{cases} p_n(x,y) & \text{if} \quad (x,y) \in G(C^\star_{n,l}), \\ 1 - \sum_{y' \in C^\star_{n,l}} p_n(x,y'), & \text{if} \quad x \in C^\star_{n,l}, y = \Delta, \\ 1, & \text{if} \quad x = y = \Delta, \\ 0, & \text{else}. \end{cases} \tag{5.4}$$

Thus $J^\star_{n,l}$ can be viewed as the restriction of J_n to $C^\star_{n,l}$, killed on the boundary $\partial C^\star_{n,l}$. We also call $Q_{n,l} = (q_{n,l}(x,y))$ the sub-Markovian restriction of $P^\star_{n,l}$ to $C^\star_{n,l}$, namely $q_{n,l} : C^\star_{n,l} \times C^\star_{n,l} \to [0,1]$,

$$q_{n,l}(x,y) = \begin{cases} p_n(x,y) & \text{if} \quad (x,y) \in G(C^\star_{n,l}), \\ 0, & \text{else}. \end{cases} \tag{5.5}$$

Then, $\Lambda^\dagger_n(j)$ in (3.12) is equal in distribution to the absorption time

$$T^\star_{n,l} = \inf\{i \in \mathbb{N} \mid J^\star_{n,l}(i) = \Delta\} \tag{5.6}$$

of the process $J^\star_{n,l}$ started in $J^\star_{n,l}(0) = J^\dagger_n(j)$. Furthermore, using the notation introduced below (4.8), we have that for all $x \in C^\star_{n,l}$ and $i > 0$,

$$\mathcal{P}_x\left(T^\star_{n,l} > i\right) = \textstyle\sum_{y \in C^\star_{n,l}} (\delta_x, Q^i_{n,l}\delta_y). \tag{5.7}$$

When $|C^\star_{n,l}| = 2$, the right hand side of (5.7) is easily worked out by hand and gives (5.2). When $|C^\star_{n,l}| > 2$, we proceed as in (4.10) - (4.13), observing that for all $1 \leq k \leq i$

$$\begin{aligned} \textstyle\sum_{x_k \in C^\star_{n,l}} q_{n,l}(x_{k-1}, x_k) &= 1 - \textstyle\sum_{x_k \notin C^\star_{n,l}} p_n(x_{k-1}, x_k) && (5.8) \\ &= 1 - \frac{(n - |C^\star_{n,l}|)}{(n - |C^\star_{n,l}|) + \sum_{x_k \in C^\star_{n,l}} e^{-\beta \max(H_n(y), H_n(x))}}, && (5.9) \\ &\leq 1 - \left[1 + \tfrac{|C^\star_{n,l}|}{n - |C^\star_{n,l}|} \varrho_{n,l}(0)\right]^{-1}, && (5.10) \end{aligned}$$

where $\varrho_{n,l}(0)$ is as in (5.1). Using this in (5.7) then yields

$$\mathcal{P}_x\left(T^\star_{n,l} > i\right) \leq \left(1 - \left[1 + \tfrac{|C^\star_{n,l}|}{n - |C^\star_{n,l}|} \varrho_{n,l}(0)\right]^{-1}\right)^i \tag{5.11}$$

which is tantamount to (5.3). The proof of the proposition is complete. □

6. Properties of the effective jump chain J°_n

This section gathers needed results on the chain J°_n. The first proposition, which is central to the strategy of Sections 8 and 9, states that J°_n is fast mixing. Given a numerical constant $0 < C < \infty$, define

$$\ell^\circ_n = \left\lceil C n^{2(c_\star + 1)} / (\log n)^2 \right\rceil. \tag{6.1}$$

Proposition 6.1. *Assume that $c_\star > 1+\log 4$. There exists $0<C<\infty$ such that the following holds. For all $\beta>0$, there exists a subset $\Omega_1\subset\Omega$ with $\mathbb{P}(\Omega_1)=1$ such that, on Ω_1, for all but a finite number of indices n, for all pairs $x\in\mathcal{V}_n^\circ$, $y\in\mathcal{V}_n^\circ$, and all $i\geq 0$,*

$$\left|P^\circ_{\pi_n^\circ}\left(J_n^\circ(i+\ell_n^\circ)=y, J_n^\circ(i)=x\right)-\pi_n^\circ(x)\pi_n^\circ(y)\right|\leq \delta_n\pi_n^\circ(x)\pi_n^\circ(y)\,, \tag{6.2}$$

where $0\leq\delta_n\leq 2^{-n}$.

Thus, the random variables $J_n^\circ(\ell_n^\circ i)$, $i\in\mathbb{N}$, are close to independent and distributed according to the invariant distribution π_n°. The next proposition provides bounds on certain mean local times that are needed to control stretches of trajectories of length ℓ_n°. Recall that $I_n^\star$ is the set of isolated vertices in the partition (6.3) and, given a constant $\kappa_\star>1$, set

$$\mathcal{W}_n=\left\{x\in\mathcal{V}_n^\circ \mid \textstyle\sum_{1\leq l\leq L^\star}|\partial x\cap\partial C^\star_{n,l}|\leq\kappa_\star, \sum_{1\leq l\leq L^\star}|\partial_2 x\cap\partial C^\star_{n,l}|\leq\frac{n}{\log n}\right\}. \tag{6.3}$$

Proposition 6.2. *Assume that $c_\star>1$. There exists a subset $\Omega^{\mathsf{SRW}}\subset\Omega$ with $\mathbb{P}(\Omega^{\mathsf{SRW}})=1$ such that, on $\Omega^{\mathsf{SRW}}\cap\Omega^\star$, for all but a finite number of indices n, the following holds: there exist constants $0<C_\circ, C'_\circ<\infty$ such that, for all $\kappa_\star>0$,*
(i) for all $z\in I_n^\star\cap\mathcal{W}_n$,

$$\sum_{l=1}^{\ell_n^\circ-1}P^\circ\left(J_n^\circ(l+2)=z\mid J_n^\circ(0)=z\right)\leq\frac{C_\circ}{\log n}, \tag{6.4}$$

(ii) for all $1\leq l\leq L^\star$ all $z\in\partial C^\star_{n,l}$ and $z'\in\partial C^\star_{n,l}\cap\mathcal{W}_n$,

$$\sum_{l=1}^{\ell_n^\circ-1}P^\circ\left(J_n^\circ(l)=z'\mid J_n^\circ(0)=z\right)\leq\frac{C'_\circ}{\log n}. \tag{6.5}$$

SRW in Ω^{SRW} above stands for Symmetric Random Walk. The reason for this will become clear from the proof (see Lemma 6.12). One may however already observe that the behavior of J_n° in Proposition 6.1 and Proposition 6.2 is reminiscent of SRW (see e.g. Section 3 of [25]) and as the next proposition shows, so is that of its invariant measure.

Proposition 6.3. *Assume that $c_\star>2$. For all $\beta>0$,*

$$\pi_n^\circ(x)=1/|\mathcal{V}_n^\circ|,\quad x\in\mathcal{V}_n^\circ, \tag{6.6}$$

where, on $\Omega^\star$, for all but a finite number of indices n,

$$|\mathcal{V}_n^\circ|=2^n\left[1-n^{-2c_\star+1}(1+\mathcal{O}(n^{-(c_\star-1)}))\right]. \tag{6.7}$$

Let us immediately give the short proof of Proposition 6.3.

Proof. Because the process X_n has a unique reversible invariant measure, $G_{\beta,n}$, the jump chain also has unique reversible invariant measure, which is the measure defined on $\mathcal{V}_n$ by

$$\pi_n(x)=\frac{\lambda_n(x)G_{\beta,n}(x)}{\sum_{x\in\mathcal{V}_n}\lambda_n(x)G_{\beta,n}(x)}=\frac{\sum_{y:(x,y)\in\mathcal{E}_n}e^{-\beta\max(H_n(y),H_n(x))}}{\sum_{x\in\mathcal{V}_n}\sum_{y:(x,y)\in\mathcal{E}_n}e^{-\beta\max(H_n(y),H_n(x))}}. \tag{6.8}$$

By this and (3.6) $\pi_n^\circ(x)=(nW^\circ_{\beta,n})^{-1}\sum_{y:(x,y)\in\mathcal{E}_n}e^{-\beta\max(H_n(y),H_n(x))}$, $x\in\mathcal{V}_n^\circ$, where $W^\circ_{\beta,n}=n^{-1}\sum_{x\in\mathcal{V}_n^\circ}\sum_{y:(x,y)\in\mathcal{E}_n}e^{-\beta\max(H_n(y),H_n(x))}$. But by (2.7) and the definition (3.1) of $\mathcal{V}_n^\circ$, $\max(H_n(y),H_n(x))=0$ whenever one of the two vertices $\{x,y\}$ lies in $\mathcal{V}_n^\circ$. Hence $W^\circ_{\beta,n}=|\mathcal{V}_n^\circ|$, yielding (6.6). Since $|\mathcal{V}_n^\circ|=2^n-\sum_{l=1}^{L^\star}\left|C^\star_{n,l}\right|$, (6.7) follows from (2.12). $\square$

Our last proposition contains a rough lower bound on hitting times at stationarity that is needed in the proof of Theorem 1.3. Write

$$T^\circ(A) \equiv \inf\{i \in \mathbb{N} \mid J_n^\circ(i) \in A\}, \quad A \subseteq \mathcal{V}_n^\circ. \tag{6.9}$$

Proposition 6.4. *Assume that $c_\star > 2$. On $\Omega^\star$, for all but a finite number of indices n, we have that for all $A \subseteq \mathcal{V}_n^\circ$ and for $I_n^\star$ as in (2.7),*

$$P_{\pi_n^\circ}^\circ\left(T^\circ(A \cap I_n^\star) > t\right) \geq (1+o(1))\exp\left(-2t|A \cap I_n^\star|/|\mathcal{V}_n^\circ|\right) - \mathcal{O}(\tfrac{1}{\log n}), \quad t > 0. \tag{6.10}$$

The rest of this section is organized as follows. The proof of Proposition 6.1 is given in Subsection 6.2, and the proofs of Proposition 6.2 and Proposition 6.4 in Subsection 6.3 and Subsection 6.4, respectively. Needed estimates on the transition probabilities of J_n° are given in Subsection 6.1.

6.1. **Estimates on the transition probabilities.** We now examine the transition probabilities (3.5) of J_n°. In what follows we denote by $G^\star(A)$ the complete graph on A with self-loops. Let $G^\circ(\mathcal{V}_n^\circ)$ be the graph with vertex set $\mathcal{V}_n^\circ$ such that (x, y) is an egde of the graph if and only if $p_n^\circ(x, y) > 0$. In view of (3.4)-(3.5),

$$G^\circ(\mathcal{V}_n^\circ) = G(\mathcal{V}_n^\circ) \bigcup \left(\bigcup_{1 \leq l \leq L^\star} G^\star(\partial C_{n,l}^\star) \right). \tag{6.11}$$

Proposition 6.5. *For all $(x, y) \in G(\mathcal{V}_n^\circ)$,*

$$p_n^\circ(x, y) = 1/n, \tag{6.12}$$

and, for all $1 \leq l \leq L^\star$ and all (x, y) in $G^\star(\partial C_{n,l}^\star)$,

$$p_n^\circ(x, y) = \frac{m_{n,l}^\star(x) m_{n,l}^\star(y)}{\sum_{z \in \partial C_{n,l}^\star} m_{n,l}^\star(z)}(1 + o(1)), \tag{6.13}$$

where $n m_{n,l}^\star(x)$ is the number of vertices of $C_{n,l}^\star$ that are are distance one from x,

$$m_{n,l}^\star(x) \equiv n^{-1}\left| \left\{y \in C_{n,l}^\star \mid \mathrm{dist}(y, x) = 1\right\} \right|, \quad x \in \partial C_{n,l}^\star. \tag{6.14}$$

Proof. Clearly, if $(x, y) \in G(\mathcal{V}_n^\circ)$, $p_n^\circ(x, y) = p_n(x, y) = 1/n$, yielding (6.12). We now turn to (6.13). Let us first state two useful a priori relations

$$p_n^\circ(x, y) = p_n^\circ(y, x) \quad \forall (x, y) \in G^\circ(\mathcal{V}_n^\circ), \tag{6.15}$$

$$m_{n,l}^\star(y) = \textstyle\sum_{x \in \partial C_{n,l}^\star} p_n^\circ(x, y) \quad \forall y \in \partial C_{n,l}^\star. \tag{6.16}$$

Eq. (6.15) is reversibility. Eq. (6.16) follows from the relation $\sum_y p_n^\circ(x, y) = 1$, (6.12), (6.15), and the definition (6.14).

Given $A \subseteq \mathcal{V}_n$ write $T(A) \equiv \inf\{i \in \mathbb{N} \mid J_n(i) \in A\}$. Also recall that for $1 \leq l \leq L^\star$, $T_{n,l}^\star \equiv \inf\{i \in \mathbb{N} \mid J_n(i) \in \partial C_{n,l}^\star\}$. Then, for all $(x, y) \in G^\star(\partial C_{n,l}^\star)$,

$$p_n^\circ(x, y) = \textstyle\sum_{z \in C_{n,l}^\star} p_n(x, z) P_z\left(J_n(T_{n,l}^\star) = y\right). \tag{6.17}$$

The next lemma establishes that the exit distribution from $C_{n,l}^\star$ is independent from the entrance point, provided that the exit probability is not too small.

Lemma 6.6. *For any two distinct vertices z and $\bar{z}$ in $C_{n,l}^\star$ and any $y \in \partial C_{n,l}^\star$,*

$$P_z\left(J_n(T_{n,l}^\star) = y\right) = (1 - \tilde{\epsilon}_n) P_{\bar{z}}\left(J_n(T_{n,l}^\star) = y\right) + \tilde{\epsilon}_n, \tag{6.18}$$

where $\tilde{\epsilon}_n \leq |\partial C_{n,l}^\star| / \varrho_{n,l}(1)$.

Proof of Lemma 6.6. Note that for any two vertices z and $\bar{z}$ in $C^\star_{n,l}$,

$$\begin{aligned} P_z\left(T^\star_{n,l} \le T(\bar{z})\right) &= \textstyle\sum_{y\in\partial C^\star_{n,l}} P_z\left(T(y) \le T(\bar{z}\cup(\partial C^\star_{n,l}))\right) \le \sum_{y\in\partial C^\star_{n,l}} \frac{\pi_n(y)}{\pi_n(z)} \quad (6.19)\\ &\le |\partial C^\star_{n,l}|\varrho^{-1}_{n,l}(1), \quad (6.20)\end{aligned}$$

where the inequality in (6.19) is reversibility. Next decompose the event $\{J_n(T^\star_{n,l}) = y\}$ according to whether $\{T(\bar{z}) \ge T^\star_{n,l}\}$ or $\{T(\bar{z}) < T^\star_{n,l}\}$: by the strong Markov property,

$$P_z\left(T(\bar{z}) < T^\star_{n,l}, J_n(T^\star_{n,l}) = y\right) = P_z\left(T(\bar{z}) < T^\star_{n,l}\right) P_{\bar{z}}\left(J_n(T^\star_{n,l}) = y\right), \tag{6.21}$$

whereas $P_z\left(T^\star_{n,l} \le T(\bar{z}), J_n(T^\star_{n,l}) = y\right) \le P_z\left(T^\star_{n,l} \le T(\bar{z})\right)$. Eq. (6.18) now follows. □

Now pick an arbitrary vertex $z^\star_{n,l} \in C^\star_{n,l}$ and denote by $\mathcal{L}^\star_{n,l}$ the exit distribution

$$\mathcal{L}^\star_{n,l}(y) = P_{z^\star_{n,l}}\left(J_n(T^\star_{n,l}) = y\right), \quad y \in \partial C^\star_{n,l}. \tag{6.22}$$

Lemma 6.7. *For all $z \in C^\star_{n,l}$ and $y \in \partial C^\star_{n,l}$*

$$P_z\left(J_n(T^\star_{n,l}) = y\right) = (1 + o(1))\mathcal{L}^\star_{n,l}(y). \tag{6.23}$$

Proof of Lemma 6.7. We readily deduce from Lemma 6.6 that if $y \in \partial C^\star_{n,l}$ is such that $\mathcal{L}^\star_{n,l}(y) \ge n\tilde{\epsilon}_n$, then $P_z\left(J_n(T^\star_{n,l}) = y\right) = (1 + o(1))\mathcal{L}^\star_{n,l}(y)$, otherwise

$$P_z\left(J_n(T^\star_{n,l}) = y\right) < (n + 1)\tilde{\epsilon}_n. \tag{6.24}$$

Let us prove by contradiction that $\mathcal{L}^\star_{n,l}(y) \ge n\tilde{\epsilon}_n$ for all $y \in \partial C^\star_{n,l}$. Assume that there exists $y \in \partial C^\star_{n,l}$ such that $\mathcal{L}^\star_{n,l}(y) < n\tilde{\epsilon}_n$. Then, by (6.24) and (6.17),

$$p^\circ_n(x, y) \le (n + 1)\tilde{\epsilon}_n \textstyle\sum_{z\in C^\star_{n,l}} p_n(x, z) = (n + 1)\tilde{\epsilon}_n m^\star_{n,l}(x). \tag{6.25}$$

Summing both sides over $x \in \partial C^\star_{n,l}$,

$$\textstyle\sum_{x\in\partial C^\star_{n,l}} p^\circ_n(x, y) \le (n + 1)\tilde{\epsilon}_n \sum_{x\in\partial C^\star_{n,l}} m^\star_{n,l}(x) \le n^5\varrho^{-1}_{n,l}(1) \ll n^{-1}. \tag{6.26}$$

However, by (6.16), $\sum_{x\in\partial C^\star_{n,l}} p^\circ_n(x, y) = m^\star_{n,l}(x) \ge n^{-1}$, which is a contradiction. □

We are now ready to conclude the proof of 6.6. By (6.17) and (6.23),

$$p^\circ_n(x, y) = m^\star_{n,l}(x)\mathcal{L}^\star_{n,l}(y)(1 + o(1)). \tag{6.27}$$

Inserting this in (6.15) and summing both sides over $x \in \partial C^\star_{n,l}$ we get

$$\mathcal{L}^\star_{n,l}(y) = \frac{m^\star_{n,l}(y)}{\sum_{x\in\partial C^\star_{n,l}} m^\star_{n,l}(x)}(1 + o(1)), \tag{6.28}$$

and inserting this in turn in (6.27) yields (6.13). The proof of the proposition is done. □

6.2. **Proof of Proposition 6.1.** Let

$$1 = \vartheta^\circ_n(0) > \vartheta^\circ_n(1) \ge \vartheta^\circ_n \cdots \ge \vartheta_{n,l}(|\mathcal{V}^\circ_n| - 1) > -1 \tag{6.29}$$

denote the eigenvalues of the matrix with entries (3.5). Set $\tau^\circ_n \equiv 1/(1 - \vartheta^\circ_n(1))$ and $\beta^\circ_n \equiv 1/(1 + \vartheta^\circ_n(|\mathcal{V}^\circ_n| - 1))$. The proof of Proposition 6.1, stated at the end of this subsection, relies the following upper bounds on τ°_n and β°_n.

Proposition 6.8. *Assume that* $c_\star > 1 + \log 4$. *For all* $\beta > 0$, *there exists a subset* $\Omega_2 \subset \Omega$ *with* $\mathbb{P}(\Omega_2) = 1$ *such that, on* Ω_2, *for all but a finite number of indices* n,

$$\tau_n^\circ \le \tfrac{1}{2} n^2 (1 + o(1)), \tag{6.30}$$

and, for some constant $0 < C < \infty$ *depending on* $c_\star$,

$$\beta_n^\circ \le C n^{2c_\star + 1} / (\log n)^2. \tag{6.31}$$

Proof of the bound (6.30). The proofs of (6.30) relies on a well known bound taken from [21] (see Proposition 1' p. 38) and expressed in terms of so-called "canonical paths". For each pair of distinct vertices $x, y \in \mathcal{V}_n^\circ$, choose a path $\gamma_{x,y}^\circ$ going from x to y in the graph $G^\circ(\mathcal{V}_n^\circ)$. Paths may have repeated vertices but a given edge appears at most once in a given path. Let $\Gamma_n^\circ = \{\gamma_{x,y}^\circ\}$ denote a collection of paths (one for each pair $\{x, y\}$). Then

$$\tau_n^\circ \le \max_e \rho_n^{-1}(e) \sum_{\gamma_{x,y}^\circ \ni e} \left|\gamma_{x,y}^\circ\right| \pi_n^\circ(x) \pi_n^\circ(y), \tag{6.32}$$

where the max is over all edges $e = \{x', y'\}$ of $G^\circ(\mathcal{V}_n^\circ)$, $\rho_n(e) \equiv \pi_{n,l}^\circ(x') p_n^\circ(x', y')$, and the summation is over all paths $\gamma_{x,y}^\circ$ in Γ_n° that pass through e. The quality of the bound (6.32) now depends on making a judicious choice of the set of paths Γ_n°. We will adopt a very clever choice made in [24] where Γ_n° is constructed using paths that remain confined to the subgraph $G(\mathcal{V}_n^\circ)$ in (6.11), and so, never use the edges of the graphs $G^\star(\partial C_{n,l}^\star)$.

• ***A choice of*** Γ_n°**.** We first construct a subset Γ_n' of paths in $G(\mathcal{V}_n)$ as follows. Given $i \in \{1, \dots n\}$, and given two vertices x and $x' \in \mathcal{V}_n$ such that $x_i \ne x_i'$, let $\gamma_{x,x'}^i$ be the path obtained by going left to right cyclically from x to x', successively flipping the disagreeing coordinates, starting from the i-th coordinate. Set $\Gamma_n^i = \left\{\gamma_{x,x'}^i, x, x' \in \mathcal{V}_n\right\}$, $1 \le i \le n$. These paths are ordered in an obvious way. Given x, x' and $\gamma_{x,x'}$, let $\overline{\gamma}_{x,x'}$ be the set of vertices visited by the path $\gamma_{x,x'}$, and let $\gamma_{x,x'}^{int} = \overline{\gamma}_{x,x'} \setminus \{x, x'\}$ be the subset of "interior" vertices. We next split the set of vertices $\mathcal{V}_n$ into *good* ones and *bad* ones. Recalling (2.7), we say that a vertex is good if it belongs to $N_n^\star$; otherwise it is bad. We say that a path γ is good if all its interior points γ^{int} are good, and that a set of paths is good if all its elements are good.

The (random) set of path Γ_n' is then constructed as follows:
(i) Consider pairs x and x' such that $\mathrm{dist}(x, x') \ge n/\log n$. If $\{\gamma_{x,x'}^i, 1 \le i \le n\}$ contains a good path, choose the first such for Γ_n'; otherwise choose $\gamma_{x,x'}^1$.
(ii) Consider pairs x and x' such that $\mathrm{dist}(x, x') < n/\log n$. If there is a good vertex $x'' \in \mathcal{V}_n$ such that $\mathrm{dist}(x, x'') \ge n/\log n$ and $\mathrm{dist}(x'', x') \ge n/\log n$, and if there are good paths, one in $\left\{\gamma_{x,x''}^i, 1 \le i \le n\right\}$ and one in $\left\{\gamma_{x'',x'}^i, 1 \le i \le n\right\}$, such that the union of these two good paths is a self avoiding path of length less than n, select this union as the path connecting x to x' in Γ_n' (notice that this is a good path); otherwise choose $\gamma_{x,x'}^1$.

The key point of this construction is that Γ_n' is almost surely good. More precisely, set $\Omega_n^{\text{GOOD}} = \{\Gamma_n' \text{ is good}\}$, $n \ge 1$, and $\Omega^{\text{GOOD}} = \liminf_{n\to\infty} \Omega_n^{\text{GOOD}}$.

Proposition 6.9 (Proposition 4.1 of [24])**.** *If* $c_\star > 1 + \log 4$ *then* $\mathbb{P}\left(\Omega^{\text{GOOD}}\right) = 1$.

The set Γ_n° is now defined as the set

$$\Gamma_n^\circ \equiv \left\{\gamma_{x,y} \in \Gamma_n',\, x, y \in \mathcal{V}_n^\circ\right\} \tag{6.33}$$

obtained from Γ_n' by removing the paths whose endpoints lie in $\cup_{1 \le l \le L^\star} C_{n,l}^\star$. Hence, on Ω^{GOOD} the paths of Γ_n° only visit vertices in $\mathcal{V}_n^\circ$ following edges of $G(\mathcal{V}_n^\circ)$. This finishes our construction of Γ_n°. Note that the paths constructed in this way have length smaller than n. Thus (6.32) yields

$$\tau_n^\circ \le (n^2/|\mathcal{V}_n^\circ|)\max_{e\in G(\mathcal{V}_n^\circ)} |\{\gamma\in\Gamma_n^\circ \mid e\in\gamma\}|\,. \tag{6.34}$$

• ***Bound on*** τ_n°**.** From now on we assume that $\omega\in\Omega^{\text{GOOD}}$ so that, for all large enough n, $\Gamma_n^\circ\equiv\Gamma_n^\circ(\omega)$ is good. In that case a bad vertex can appear only at the ends of any path. Let us write

$$\tau_n^\circ = (n^2/|\mathcal{V}_n^\circ|)(\tau_n^1+\tau_n^2)\,, \tag{6.35}$$

where τ_n^1, repectively τ_n^2, is obtained by restricting the sum in (6.34) to paths connecting vertices at distance $n/\log n$ or more apart, respectively, less than $n/\log n$ apart.

On the one hand it is well known that (see e.g. Example 2.2, p. 45 in [21])

$$\tau_n^1 \le 2^{n-1}. \tag{6.36}$$

On the other hand, arguing as in [24] (see Subsection 4.2.2, page 934) that the sum in τ_n^2 is over a set of paths that connect vertices in a hypercube of dimension at most $n/\log n$ around e, we have

$$\tau_n^2 \le 2^{2n/\log n}. \tag{6.37}$$

Plugging (6.36) and (6.37) in (6.35), and using (6.7) of Proposition 6.3 to bound $|\mathcal{V}_n^\circ|$, we we get that on $\Omega^{\text{GOOD}}\cap\Omega^\star$, for large enough n,

$$\tau_n^\circ \le n^2 2^{-n}\left[1-n^{-2c_\star+1}(1+o(1))\right]^{-1}\left(2^{n-1}+2^{2n/\log n}\right)\le (n^2/2)(1+o(1)). \tag{6.38}$$

which is the upper bound (6.30) on τ_n°. $\square$

Proof of the bound (6.31). Keeping (6.11) in mind, let γ_x° be a path in $G^\circ(\mathcal{V}_n^\circ)$ from x to x with an *odd number* of edges. Since J_n° is an irreducible and aperiodic chain such paths exist. Let Γ_n^{ODD} be a collection of paths, one for each $x\in\mathcal{V}_n^\circ$. For $\rho_n(e)$ as in (6.32) define the *path length*, $|\gamma_x^\circ|$, through $|\gamma_x^\circ|=\sum_{e\in\gamma_x^\circ}\rho_n^{-1}(e)$. Then, by Proposition 2 of [21],

$$\beta_n^\circ \le \tfrac12 \max_e \sum_{\gamma_x^\circ\ni e}|\gamma_x^\circ|\,\pi_n^\circ(x), \tag{6.39}$$

where the max is over all edges $e=\{x',y'\}$ of $G^\circ(\mathcal{V}_n^\circ)$, and the summation is over all paths γ_x° in Γ_n° that pass through e. As for (6.32), the accuracy of this bound depends on how good a set of paths Γ_n^{ODD} we can find. Note that since the graph $G(\mathcal{V}_n)$ is bipartite, paths from x to x confined to the subgraph $G(\mathcal{V}_n^\circ)=G^\circ(\mathcal{V}_n^\circ)\cap G(\mathcal{V}_n)$ have an even number of edges. Thus paths with an odd number of edges must step across one of the components $C_{n,l}^\star$, that is, must use an edge of $G^\star(\partial C_{n,l}^\star)$. In order to construct such paths we first show that each vertex $x\in\mathcal{V}_n^\circ$ lies within a small distance (how small depending on $c_\star$) of some $C_{n,l}^\star$ of size two. More precisely, denoting by $\mathcal{B}_\sigma(x)=\{y\in\mathcal{V}_n\mid \operatorname{dist}(x,y)\le\sigma\}$ the ball of radius $\sigma>0$ centered at $x\in\mathcal{V}_n$ we show that:

Lemma 6.10. *If* $\sigma>2c_\star+5$ *and* $c_\star>1$ *then*

$$\mathbb{P}\left(\forall_{x\in\mathcal{V}_n}\exists_{1\le l\le L^\star:|C_{n,l}^\star|=2}\left\{C_{n,l}^\star\cap\mathcal{B}_\sigma(x)\neq\emptyset\right\}\right)\ge 1-e^{-n}. \tag{6.40}$$

Proof of Lemma 6.10. Let $\mathcal{G}_2$ be the set of undirected edges of $G(\mathcal{V}_n)$ and, for each $\{\bar x,\bar y\}$ in $\mathcal{G}_2$, define the variable $Z_n(\bar x,\bar y)\equiv\chi_n(\bar x)\chi_n(\bar y)\prod_{z\in(\partial\bar x\cup\partial\bar y)\setminus\{\bar x,\bar y\}}(1-\chi_n(z))$ where $\chi_n(\bar x)\equiv\mathbb{1}_{\{w_n(\bar x)\ge r_n(\rho_n^\star)\}}$. Note that $Z_n(\bar x,\bar y)$ is a Bernoulli r.v. with $\mathbb{P}(Z_n(\bar x,\bar y)=1)=1-\mathbb{P}(Z_n(\bar x,\bar y)=0)=n^{-2c_\star}(1-n^{-c_\star})^{2(n-1)}\equiv p_n$. By (2.7), $\{\bar x,\bar y\}$ is a connected component of size two if and only if $Z_n(\bar x,\bar y)=1$. Thus, the total number of such components intersecting the ball $\mathcal{B}_\sigma(x)$ is

$$S_{n,\sigma}(x)=\sum_{\{\bar x,\bar y\}\in\mathcal{G}_2\,:\,\{\bar x,\bar y\}\cap\in\mathcal{B}_\sigma(x)\neq\emptyset}Z_n(\bar x,\bar y), \tag{6.41}$$

and the intersection in (6.40) is non empty if and only if $S_{n,\sigma}(x) > 1$. Let us thus evaluate $S_{n,\sigma}(x)$. Clearly, this is a sum of dependent random variables. To cope with this difficulty we split it into disjoint sums as follows. Let $\mathbf{1}$ denote the vertex of $\mathcal{V}_n$ all of whose coordinates are 1 and set $\mathcal{V}_n^+ \equiv \{x \in \mathcal{V}_n \mid \mathrm{dist}(\mathbf{1}, x) = 2m, m \geq 0\}$. Then $\mathcal{G}_2 = \cup_{1\leq j\leq n}\mathcal{G}_2^j$ where, for each $1 \leq j \leq n$, $\mathcal{G}_2^j \equiv \{\{x, y\} \in \mathcal{G}_2 \mid x \in \mathcal{V}_n^+, x_j = -y_j\}$ is the set of neighboring vertices that differ in exactly the j-th coordinate. It is not hard to see that there exists a covering $\mathcal{V}_n^+ = \cup_{1\leq i\leq v_n}\mathcal{V}_n^{+,i}$ of $\mathcal{V}_n^+$ by disjoint subsets, $\mathcal{V}_n^{+,i}$, with the property that $\mathrm{dist}(x, x') \geq 6$ for all pairs x and x' in $\mathcal{V}_n^{+,i}$ and all $i \leq v_n$, where $v_n < 2n^4$, and such that $|\mathcal{V}_n^{+,i} \cap \mathcal{B}_\sigma(x)| \sim n^\sigma/v_n$ for $\sigma > 6$. Using this covering, subdivide each $\mathcal{G}_2^j$ into v_n disjoint sets,

$$\mathcal{G}_2^{j,i} \equiv \left\{\{x, y\} \in \mathcal{G}_2 \mid x \in \mathcal{V}_n^{+,i}, x_j = -y_j\right\}, \quad i = 1, \dots, v_n. \tag{6.42}$$

Then $\mathcal{G}_2^j = \cup_{1\leq i\leq v_n}\mathcal{G}_2^{j,i}$ and

$$S_{n,\sigma}(x) = \textstyle\sum_{i=1}^{v_n}\sum_{j=1}^{n} S_{n,\sigma}^{j,i}(x),\ S_{n,\sigma}^{j,i}(x) \equiv \sum_{\{\bar{x},\bar{y}\}\in\mathcal{G}_2^{j,i}\,:\,\bar{x},\bar{y}\in\mathcal{B}_\sigma(x)} Z_n(\bar{x}, \bar{y}). \tag{6.43}$$

Each $S_{n,\sigma}^{j,i}(x)$ now is a sum of independent Bernoulli r.v.'s, and can be controlled using a classical concentration bound (see e.g. [9]), yielding

$$\mathbb{P}\left(\exists_{x\in\mathcal{V}_n}\exists_{1\leq i\leq v_n}\exists_{1\leq j\leq n}\left\{\left|S_{n,\sigma}^{j,i}(x) - p_n N_n^i\right| > \sqrt{4ntN_n^i p_n(1 - p_n)}\right\}\right) \leq n v_n 2^n e^{-nt} \tag{6.44}$$

for all $t > 0$, provided that $N_n^i p_n(1-p_n) > 4nt$, where $N_n^i = |\mathcal{V}_n^{+,i} \cap \mathcal{B}_\sigma(x)|$ is the number of terms in each $S_{n,\sigma}^{j,i}(x)$. Since $N_n^i = \mathcal{O}(n^\sigma/v_n)$, the latter condition is verified whenever $\sigma - 4 - 2c_\star > 1$ and $c_\star > 1$. In that case $p_n N_n^i \geq \mathcal{O}(n^{\sigma-4-2c_\star}) > \mathcal{O}(n) \gg 1$, and so $p_n N_n^i - \sqrt{nN_n^i p_n(1 - p_n)} \gg 1$ for all large enough n. Choosing $t = 2$ in (6.44) then yields the claim of the lemma. $\square$

• ***A choice of*** Γ_n^{ODD}**.** We are now ready to construct the set of paths $\Gamma_n^{\text{ODD}} = \{\gamma_x^\circ, x \in \mathcal{V}_n^\circ\}$. The notations and definitions introduced in the paragraph below (6.33) (for the construction of the set Γ_n) are used in several places but not always reminded.

Assume from now on that $\sigma > 2c_\star + 5$ and $c_\star > 1$. By Lemma 6.10 and Borel-Cantelli Lemma, there exists a subset $\Omega_2' \subset \Omega$ with $\mathbb{P}(\Omega_2') = 1$ such that, on Ω_2', for all but a finite number of indices n, each ball $\mathcal{B}_\sigma(x)$ contains at least one vertex that belongs to a connected component $C_{n,l}^\star$ of size two. Given $x \in \mathcal{V}_n^\circ$ let $y \in \mathcal{B}_\sigma(x)$ be any such vertex (how to choose y will be specified later), and denote by $D(x, y)$ the set of coordinates where x and y disagree. In order to construct the path $\gamma_x^\circ \in \Gamma_n^{\text{ODD}}$ we first construct a collection $\widetilde{\Gamma}_n(x, y) = \left\{\tilde{\gamma}_{x,y}^i, i \in \{1, \dots n\} \setminus D(x, y)\right\}$ of $n - |D(x, y)|$ paths going from x to y as follows. Given $i \in \{1, \dots n\} \setminus D(x, y)$, let x^i and y^i be, respectively, the vertices obtained from x and y by flipping their i-th coordinate. Note that $D(x, y) = D(x^i, y^i)$ and recall that γ_{x^i,y^i}^1 denotes the path that goes left to right cyclically from x^i to y^i, successively flipping the disagreeing coordinates in $D(x^i, y^i)$, starting from the first. We then define $\tilde{\gamma}_{x,y}^i$ as the path that first steps from x to x^i, follows the path γ_{x^i,y^i}^1 from x^i to y^i, and takes a final step from y^i to y. Clearly, $\widetilde{\Gamma}_n(x, y)$ forms a collection of $n - |D(x, y)|$ interior disjoint paths of length $|D(x, y)|$. Let us show that almost surely, each $\widetilde{\Gamma}_n(x, y)$ contains at least $\lfloor \log n \rfloor$ good paths. For this set $\kappa \equiv \kappa(n) = \lfloor \log n \rfloor$, and define

$\Omega_n^{\text{ODD}}(x, y) = \{\exists i_1 \neq \cdots \neq i_\kappa \in \{1, \dots n\} \setminus D(x, y) \mid \tilde{\gamma}_{x,y}^{i_j}$ is good for each $1 \leq j \leq \kappa\}$, $\Omega_n^{\text{ODD}} = \bigcap_{x\in\mathcal{V}_n^\circ}\bigcap_{y\in\mathcal{B}_\sigma(x)} \Omega_n^{\text{ODD}}(x, y)$, and $\Omega^{\text{ODD}} = \liminf_{n\to\infty} \Omega_n^{\text{ODD}}$. We then have:

Lemma 6.11. $\mathbb{P}(\Omega^{\text{ODD}} \mid \Omega_2') = 1$.

Proof of Lemma 6.11. Fix a realization $\omega \in \Omega_2'$ of the random environment and consider $\widetilde{\Gamma}_n(x,y)$. By construction, there exists $j \in \{1, \dots n\}$ such that $\{y, y^j\} = C^\star_{n,l}$ for some l. Hence $\tilde{\gamma}^j_{x,y}$ is bad. Consider now $\widetilde{\Gamma}_n(x,y) \setminus \{\tilde{\gamma}^j_{x,y}\}$. Clearly, this set forms a collection of $n - |D(x,y)| - 1$ interior disjoint paths of length $3 \le |D(x,y)| + 2 \le \sigma + 2$.The probability for a given vertex to be bad is $n^{-c_\star}$. Thus, the probability of a given path not to be good is at most $(\sigma+2)n^{-c_\star}$ and, for any given k-tuple $\{i_1, \dots, i_\kappa\}$, $\mathbb{P}(\exists_{1\le j\le\kappa} \tilde{\gamma}^{i_j}_{x,y} \text{ is not good}) \le \kappa(\sigma+2)n^{-c_\star}$. Since there are at least $n - |D(x,y)| - 1 \ge n - \sigma - 1$ interior disjoint paths, there are at least $\lceil (n-\sigma-1)/\kappa \rceil$ mutually disjoint κ-tuples of such paths, two κ-tuples being disjoint if $\{i_1, \dots, i_\kappa\} \cap \{i_1', \dots, i_\kappa'\} = \emptyset$. By independence, $1 - \mathbb{P}(\Omega_n^{\text{ODD}}(x,y) \mid \Omega_2') \le (\kappa(\sigma+2)n^{-c_\star})^{\lceil (n-\sigma-1)/\kappa \rceil}$. Thus $1 - \mathbb{P}(\Omega_n^{\text{ODD}} \mid \Omega_2') \le n^\sigma 2^n (\kappa(\sigma+2)n^{-c_\star})^{\lceil (n-\sigma-1)/\kappa \rceil}$, and since for $\kappa = \lfloor \log n \rfloor$ this is summable, the claim of the lemma follows from Borel-Cantelli Lemma. □

On $\Omega^{\text{ODD}} \cap \Omega_2'$ we construct the path γ_x°, using $\widetilde{\Gamma}_n(x,y)$, as follows. Take any two good paths in $\widetilde{\Gamma}_n(x,y)$, say $\tilde{\gamma}^{i_1}_{x,y}$ and $\tilde{\gamma}^{i_2}_{x,y}$. These paths have equal number of edges, $|D(x,y)| + 1$, and enter ∂y at the vertices y^{i_1} and y^{i_2}, respectively. Because y belongs to a connected component $C^\star_{n,l}$ of size two, $\{y^{i_1}, y^{i_2}\}$ is an edge of the associated complete graph $G^\star(\partial C^\star_{n,l})$ in (6.11). We then define γ_x° as the path that goes from x to y^{i_1} along the edges of $\tilde{\gamma}^{i_1}_{x,y}$ (in $|D(x,y)|$ steps), traverses $C^\star_{n,l}$ along the edge $\{y^{i_1}, y^{i_2}\}$ (in one step), and goes from y^{i_2} to x travelling backwards along the edges of $\tilde{\gamma}^{i_2}_{x,y}$ (in again $|D(x,y)|$ steps). Thus γ_x° is a path in $G^\circ(\mathcal{V}_n^\circ)$ from x to x with $2|D(x,y)| + 1$ edges.

We still have to specify how to choose the vertex y in the above construction, as well as the two good paths in $\widetilde{\Gamma}_n(x,y)$. Note first that by (6.44), for each $x \in \mathcal{V}_n^\circ$, the ball $\mathcal{B}_\sigma(x)$ contains $np_n \sum_{1\le i\le v_n} N_n^i(1+o(1)) = \mathcal{O}(n|\mathcal{B}_\sigma(x)|/n^{2c_\star})$ occupied vertices, y, belonging to distinct connected components, $C^\star_{n,l}$, of size two. We may and will choose the pairs (x,y) in such a way that each of these components $C^\star_{n,l}$ is traversed by $\mathcal{O}(|\mathcal{B}_\sigma(0)|/\left(np_n \sum_{1\le i\le v_n} N_n^i(1+o(1))\right) = \mathcal{O}(n^{2c_\star-1})$ paths γ_x° connecting that $C^\star_{n,l}$ to vertices x at distance at most σ from it. Next, by Lemma 6.11, for each pair (x,y) as above there are at least $\kappa = \lfloor \log n \rfloor$ good paths in $\widetilde{\Gamma}_n(x,y)$, and so, there are at least $\kappa(\kappa-1)/2$ ways to choose the edge of the complete graph $G^\star(\partial C^\star_{n,l})$ through which γ_x° steps across $C^\star_{n,l}$. Therefore, we may and will choose these two good paths in such a way that each of the $(n-1)(n-2)/2$ edges of $G^\star(\partial C^\star_{n,l})$ connecting these two good paths is traversed by at most a fraction $2/\kappa(\kappa-1)$ of the total number of paths that cross $C^\star_{n,l}$ (this can probably be improved but not easily since for a given y the set $\{i_1, \dots, i_\kappa\}$ of good paths's indices generated by different pairs (x,y) are not independent subsets of $\{1, \dots, n\}$).

Our construction of Γ_n^{ODD} is now completed. Observe that it guarantees that there are at most $\mathcal{O}(n^{2c_\star-1}/(\log n)^2)$ paths in Γ_n^{ODD} that contain a given edge in any given $G^\star(\partial C^\star_{n,l})$.

• ***Bound on*** β_n°**.** Assume that $c_\star > 1$ and take $\sigma > 2c_\star + 5$ in (6.40). Then, on $\Omega^{\text{ODD}} \cap \Omega_2'$, for all large enough n, paths in Γ_n^{ODD} have at most 2σ edges, and by (6.6) and (6.12)-(6.13) of Proposition 6.5,

$$|\gamma_x^\circ| = \sum_{e\in\gamma_x^\circ} \rho_n^{-1}(e) \le (\pi_n^\circ(x))^{-1} \left[4n\sigma + 2n(n-1)(1+o(1))\right]. \tag{6.45}$$

Furthermore, by construction,

$$\max_e \sum_{\gamma_x^\circ \ni e} 1 \le \max_{1\le l\le L^\star : |C^\star_{n,l}|=2} \; \max_{e\in G^\star(\partial C^\star_{n,l})} \sum_{\gamma_x^\circ\ni e} 1 \le \mathcal{O}(n^{2c_\star-1}/(\log n)^2). \tag{6.46}$$

Inserting the last two bounds in (6.39) yields the upper bound (6.31) on β_n°.

Taking $\Omega_2 = \Omega^\star \cap \Omega^{\mathsf{GOOD}} \cap \Omega^{\mathsf{ODD}} \cap \Omega_2'$ concludes the proof of Proposition 6.8. $\square$

Proof of Proposition 6.1. By (1.9) of Proposition 3 of [21], for all $x \in \mathcal{V}_n^\circ$ and all $l \in \mathbb{N}$,

$$\left\| P_x^\circ \left(J_n^\circ(l) = \cdot \right) - \pi_n^\circ(\cdot) \right\|_{TV} \leq \left(\tfrac{1-\pi_n^\circ(x)}{4\pi_n^\circ(x)} \right)^{1/2} \left(\max\left\{ 1 - \tfrac{1}{\tau_n^\circ}, 1 - \tfrac{1}{\beta_n^\circ} \right\} \right)^l \tag{6.47}$$

where τ_n° and β_n° are defined below (6.29). From this, Proposition 6.8, and Proposition 6.3, it follows that if $c_\star > 1 + \log 4$ then, on $\Omega_1 \equiv \Omega^\star \cap \Omega_2$, for all n large enough, for all pairs $x \in \mathcal{V}_n^\circ, y \in \mathcal{V}_n^\circ$ and all $i \geq 0$, taking ℓ_n° as in (6.1), $|P_x^\circ\left(J_n^\circ(i + \ell_n^\circ) = y\right) - \pi_n^\circ(y)| \leq \delta_n \pi_n^\circ(y)$ for some $0 \leq \delta_n \leq 2^{-n}$ provided that the constant C in (6.1) is chosen big enough. The proof of Proposition 6.1 is done. $\square$

6.3. **Mean local times: proof of Proposition 6.2.** Let J_n^{SRW} and P^{SRW} denote, respectively, the symmetric random walk on $\mathcal{V}_n$ (hereafter SRW) and its law. More precisely,

$$p_n^{\mathsf{SRW}}(x,y) \equiv P^{\mathsf{SRW}}\left(J_n^{\mathsf{SRW}}(1) = y \mid J_n^{\mathsf{SRW}}(1) = x\right) = \begin{cases} \frac{1}{n} & \text{if } \mathrm{dist}(x,y) = 1, \\ 0, & \text{else.} \end{cases} \tag{6.48}$$

We also write P_x^{SRW} for the law of J_n^{SRW} started in x. The proof of Proposition 6.2 relies on three key properties of J_n^{SRW} that we state below in the form of three lemmata.

Our first lemma provides an estimate on the hitting time of the set $\mathcal{V}_n^\star \equiv \cup_{1 \leq l \leq L^\star} C_{n,l}^\star$. For $A \subset \mathcal{V}_n$ let $T^{\mathsf{SRW}}(A)$ be the hitting time

$$T^{\mathsf{SRW}}(A) = \inf\left\{ k \in \mathbb{N} : J_n^{\mathsf{SRW}}(k) \in A \right\}. \tag{6.49}$$

Lemma 6.12. *Assume that $c_\star > 1$. There exists a subset $\Omega^{\mathsf{SRW}} \subset \Omega$ with $\mathbb{P}\left(\Omega^{\mathsf{SRW}}\right) = 1$ such that, on Ω^{SRW}, for all but a finite number of indices n the following holds: for all sequences $l_n > 0$ such that $l_n/n^{2c_\star} \leq C$ for some constant $0 < C < \infty$,*

$$\max_{x \in \mathcal{V}_n} \left| P_x^{\mathsf{SRW}}\left(T^{\mathsf{SRW}}(\mathcal{V}_n^\star \setminus x) \geq l_n\right) - e^{-l_n/n^{2c_\star}} \right| \leq C' \left[\frac{1}{n} + \frac{1}{\log n^{2c_\star}} + \frac{n \log n}{n^{2c_\star}} \right], \tag{6.50}$$

where $0 < C' < \infty$ is a numerical constant.

Proof of Lemma 6.12. This is proved using Theorem 1.3 of [16], proceeding in the same way as in Theorem 1.1 of [16] on the hitting time of so-called percolation clouds. (In particular, one proceeds as in (6.42)-(6.43) to extract sums of independent Bernoulli random variables in the verification of the conditions of Theorem 1.3 of [16].) $\square$

The next two lemmata bound the mean number of returns to a given vertex, z, respectively the mean local time in z, in a time interval of the form $\{3, \dots, m\}$, $m \leq \lceil n^{2(c_\star+1)} \rceil$.

Lemma 6.13. *For all $m \leq \lceil n^{2(c_\star+1)} \rceil$, all $z \in \mathcal{V}_n$, and $a \in \{0,1,2,3\}$,*

$$\sum_{l=1}^{m} P_z^{\mathsf{SRW}}\left(J_n^{\mathsf{SRW}}(l+a) = z\right) \leq \frac{c}{n^b}, \quad b = \begin{cases} 1, & \text{if } a \in \{0,1\} \\ 2, & \text{if } a \in \{2,3\} \end{cases}, \tag{6.51}$$

where $0 < c < \infty$ is a numerical constant.

Proof. The lemma is proved in exactly the same way as Proposition 3.2 of [25]. $\square$

Lemma 6.14. *For all $m \leq \lceil n^{2(c_\star+1)} \rceil$ and all y, z such that $\mathrm{dist}(y,z) = d \geq 1$,*

$$\sum_{l=1}^{m} P_y^{\mathsf{SRW}}\left(J_n^{\mathsf{SRW}}(l) = z\right) \leq \frac{c'}{n^d} \mathbb{1}_{d \leq 4} + \frac{c'}{n^4} \mathbb{1}_{d \geq 5}, \tag{6.52}$$

where $0 < c' < \infty$ is a numerical constant.

Proof of Lemma 6.14. The proof draws on the results of [7] where a d'-dimensional version of the Ehrenfest scheme, called lumping, was introduced and analyzed (hereafter and whenever possible we use the notations of [7]). Without loss of generality we may take $y \equiv 1$ to be the vertex all of whose coordinates take the value 1. Let γ^Λ be the map (1.7) of [7] derived from the partition of $\Lambda \equiv \{1, \dots, n\}$ into $d' = 2$ classes, $\Lambda = \Lambda_1 \cup \Lambda_2$, defined through the relation: $i \in \Lambda_1$ if the i^{th} coordinate of z is 1, and $i \in \Lambda_2$ otherwise. The resulting lumped chain, $X_n^\Lambda \equiv \gamma^\Lambda(J_n^{\mathsf{SRW}})$, has range $\Gamma_{n,2} = \gamma^\Lambda(\mathcal{V}_n) \subset [-1,1]^2$. Note that the vertices y and z of $\mathcal{V}_n$ are mapped, respectively, onto the corners $1 \equiv (1,1)$ and $x \equiv (1,-1)$ of $[-1,1]^2$. Denoting by $\mathbb{P}^\Lambda$ the law of X_n^Λ, we have,

$$P_y^{\mathsf{SRW}}(J_n^{\mathsf{SRW}}(l) = z) = \mathbb{P}^\Lambda(X_n^\Lambda(l) = x \mid X_n^\Lambda(0) = 1). \tag{6.53}$$

Write $\tau_x^{x'} = \inf\{k > 0 \mid X_n^\Lambda(0) = x', X_n^\Lambda(k) = x\}$. Without loss of generality we may assume that $0 \in \Gamma_{n,2}$ (namely, both Λ_1 and Λ_2 have even cardinality). Then, decomposing (6.53) according to whether, starting from 1, X_n^Λ visits 0 before it visits x or not, we get: $\mathbb{P}^\Lambda(X_n^\Lambda(l) = x \mid X_n^\Lambda(0) = 1) = A + B$,

$$A = \mathbb{P}^\Lambda(X_n^\Lambda(l) = x, \tau_0^1 < \tau_x^1), \tag{6.54}$$

$$B = \mathbb{P}^\Lambda(X_n^\Lambda(l) = x, \tau_0^1 \geq \tau_x^1). \tag{6.55}$$

By Theorem 3.2 of [7], for all y, z such that $\mathrm{dist}(z,y) \geq d$,

$$B \leq \mathbb{P}^\Lambda(\tau_x^1 \leq \tau_0^1) \leq F_{n,2}(\mathrm{dist}(z,y)) \leq c_1(n^{-d}\mathbb{1}_{d\leq 4} + n^{-d^*}\mathbb{1}_{d\geq 5}) \tag{6.56}$$

where $d^* = \frac{d+4}{2}$ if d is even, $d^* = \frac{d+3}{2}$ if d is odd, and $0 < c_1 < \infty$ is a constant. Of course $A = 0$ for all l such that $l < n/2$ since the chain X_n^Λ needs at least $n/2$ steps to travel from the vertex 1 to 0. To bound A when $l \geq n/2$ we condition on the time of the last visit to 0 before time l, and bound the probability of the latter event by 1. This yields

$$A \leq l\mathbb{P}^\Lambda(\tau_x^0 < \tau_0^0) = l\tfrac{\mathbb{Q}_n(x)}{\mathbb{Q}_n(0)}\mathbb{P}^\Lambda(\tau_0^x < \tau_x^x) \leq l\tfrac{\mathbb{Q}_n(x)}{\mathbb{Q}_n(0)}, \tag{6.57}$$

where the equality in the middle is reversibility, and where $\mathbb{Q}_n$, defined in Lemma 2.2 of [7], denotes the invariant measure of X_n^Λ. We are thus left to estimate the ratio of invariant masses in (6.57). By (2.4) of [7] we get that $\frac{\mathbb{Q}_n(x)}{\mathbb{Q}_n(0)} \leq |\{x' \in \mathcal{V}_n \mid \gamma^\Lambda(x') = 0\}|^{-1} \leq e^{-c_2 n}$ for some constant $0 < c_2 < \infty$. Gathering our bounds we get that for all y, z such that $\mathrm{dist}(y,z) \geq 4\lfloor c_\star \rfloor + 3$,

$$P_y^{\mathsf{SRW}}(J_n^{\mathsf{SRW}}(l) = z) = A + B \leq c_1 n^{-2(\lfloor c_\star \rfloor+1)-1} + l e^{-c_2 n} \leq c_3 n^{-2(\lfloor c_\star \rfloor+1)-1} \tag{6.58}$$

for some constant $0 < c_3 < \infty$, so that for all $m \leq \lceil n^{2(c_\star+1)} \rceil$,

$$\textstyle\sum_{l=1}^m P_y^{\mathsf{SRW}}(J_n^{\mathsf{SRW}}(l) = z) \leq c_3 n^{-1}. \tag{6.59}$$

It remains to treat the cases $1 \leq \mathrm{dist}(y,z) \leq 4\lfloor c_\star \rfloor + 2$. To this end consider the event $\mathcal{A}_z \equiv \{\forall i \leq l \, \mathrm{dist}(J_n^{\mathsf{SRW}}(i), z) < 4\lfloor c_\star \rfloor + 3\}$. Decomposing its complement, $\mathcal{A}_z^c$, on the place and time of the first visit of the chain to the ball of radius $4\lfloor c_\star \rfloor + 3$, we get by the Markov property and (6.58) that

$$P_y^{\mathsf{SRW}}(J_n^{\mathsf{SRW}}(l) = z, \mathcal{A}_z^c) \leq c_3 n^{-2(\lfloor c_\star \rfloor+1)-1}. \tag{6.60}$$

Next, by reversibility (the invariant measure of J_n^{SRW} being the uniform measure),

$$P_y^{\mathsf{SRW}}(J_n^{\mathsf{SRW}}(l) = z, \mathcal{A}_z) = P_z^{\mathsf{SRW}}(J_n^{\mathsf{SRW}}(l) = y, \mathcal{A}_z) \leq P_z^{\mathsf{SRW}}(\mathcal{A}_z). \tag{6.61}$$

Let us thus estimate the probability $P_z^{\mathsf{SRW}}(\mathcal{A}_z)$ that starting in z, the chain did not exit a ball of radius $4\lfloor c_\star\rfloor+2$ centered at z by time l. This means that every step it takes, the chain flips a coordinate of z in such a way that the total number of coordinates of z and $J_n^{\mathsf{SRW}}(i)$ that disagree is at most $4\lfloor c_\star\rfloor+2$ for each $i\le l$. If $l\ge 4\lfloor c_\star\rfloor+2$, this implies that $(l-4\lfloor c_\star\rfloor+2)/2$ of its l steps (respectively, $(l-4\lfloor c_\star\rfloor+2+1)/2$ of them) consist in flipping back a coordinate to its initial position if $l-4\lfloor c_\star\rfloor+2$ is even (respectively, if $l-4\lfloor c_\star\rfloor+2$ is odd). Each time such a backward flip occurs the chain chooses one in at most $4\lfloor c_\star\rfloor+2$ flipped coordinates. Thus, for all $l\ge 4\lfloor c_\star\rfloor+2$,

$$P_y^{\mathsf{SRW}}(\mathcal{A}_z)\le((4\lfloor c_\star\rfloor+2)/n)^{\frac{l-(4\lfloor c_\star\rfloor+2)}{2}}\mathbb{1}_{l\,\mathrm{even}}+((4\lfloor c_\star\rfloor+2)/n)^{\frac{l-(4\lfloor c_\star\rfloor+1)}{2}}\mathbb{1}_{l\,\mathrm{odd}}. \tag{6.62}$$

Plugging (6.62) in (6.61) yields that for all y,z such that $1\le\mathrm{dist}(y,z)\le 4\lfloor c_\star\rfloor+2$,

$$\textstyle\sum_{l=4\lfloor c_\star\rfloor+3}^{m}P_y^{\mathsf{SRW}}(J_n^{\mathsf{SRW}}(l)=z,\mathcal{A}_z)\le c_4n^{-1}, \tag{6.63}$$

for all $m\le\lceil n^9\rceil$ and some constant $0<c_4<\infty$, while by simple combinatorics,

$$\textstyle\sum_{l=1}^{4\lfloor c_\star\rfloor+2}P_y^{\mathsf{SRW}}(J_n^{\mathsf{SRW}}(l)=z,\mathcal{A}_z)\le\sum_{l=1}^{4\lfloor c_\star\rfloor+2}P_y^{\mathsf{SRW}}(J_n^{\mathsf{SRW}}(l)=z)\le c_5n^{-1}, \tag{6.64}$$

for some $0<c_5<\infty$. Combining (6.59), (6.63) and (6.64) finishes the proof. □

We are now ready to give the proof of Proposition 6.2.

Proof of Proposition 6.2, (i). Given $y\in\mathcal{V}_n$ denote respectively by P_y°, P_y, and P_y^{SRW} the laws of J_n°, J_n, and J_n^{SRW} started in y. The idea behing the proof is to decompose the paths of J_n at visits to the set $\mathcal{V}_n^\star\equiv\cup_{1\le l\le L^\star}C_{n,l}^\star$, and use that, away from this set, J_n reduces to SRW. To this end recall (6.49) and set

$$T_n^{\mathsf{SRW},\star}\equiv\inf\left\{k\in\mathbb{N}:J_n^{\mathsf{SRW}}(k)\in\mathcal{V}_n^\star\right\}, \tag{6.65}$$
$$T_n^\star\equiv\inf\left\{i\in\mathbb{N}\mid J_n(i)\in\mathcal{V}_n^\star\right\}. \tag{6.66}$$

Let $z\in I_n^\star$ be fixed. Since by definition $J_n^\circ(i)\equiv J_n(T_{n,i}^\circ)$, we may write

$$\textstyle\sum_{k=1}^{\ell_n^\circ-1}P_z^\circ(J_n^\circ(k+2)=z)=\sum_{k=1}^{\ell_n^\circ-1}P_z\left(J_n(T_{n,k+2}^\circ)=z\right)=I_1+I_2 \tag{6.67}$$

where

$$I_1\equiv\textstyle\sum_{k=1}^{\ell_n^\circ-1}P_z\left(J_n(T_{n,k+2}^\circ)=z,T_n^\star>k+2\right), \tag{6.68}$$
$$I_2\equiv\textstyle\sum_{k=1}^{\ell_n^\circ-1}P_z\left(J_n(T_{n,k+2}^\circ)=z,T_n^\star\le k+2\right). \tag{6.69}$$

In view of (3.3)-(3.4), $T_{n,i}^\circ=i$ for all $i\in\{0,\dots,T_n^\star-1\}$. Hence

$$I_1=\textstyle\sum_{k=1}^{\ell_n^\circ-1}P_z\left(J_n(k+2)=z,T_n^\star>k+2\right), \tag{6.70}$$

and since up to time $T_n^\star$ the transition probabilities of J_n are those of SRW,

$$I_1\le\textstyle\sum_{k=1}^{\ell_n^\circ-1}P_z^{\mathsf{SRW}}(J_n^{\mathsf{SRW}}(k+2)=z)\le cn^{-2}, \tag{6.71}$$

where the last inequality is (6.51).

To Bound I_2 note that the event $\{T_n^\star\le k+2\}$ can be written as the disjoint union

$$\{T_n^\star\le k+2\}=\cup_{i\le k+2}\cup_{y\in\mathcal{V}_n^\star}\{T_n^\star=i,J_n(T_n^\star)=y\}. \tag{6.72}$$

Thus

$$I_2=\textstyle\sum_{k=1}^{\ell_n^\circ-1}\sum_{i=1}^{k+2}\sum_{y\in\mathcal{V}_n^\star}P_z\left(J_n(T_{n,k+2}^\circ)=z,T_n^\star=i,J_n(T_n^\star)=y\right). \tag{6.73}$$

As above note that $T^\circ_{n,i} = i$ for all $i \in \{0, \dots, T^\star_n - 1 = i-1\}$, that $T^\star_n = T^\circ_{n,i-1} + 1$, and that in the time interval $\{0, \dots, T^\star_n\}$, J_n has the same transition probabilities as SRW. By this and the Markov property, the probability in (6.73) is equal to

$$P^{\mathsf{SRW}}_z \left(T^{\mathsf{SRW},\star}_n = i, J^{\mathsf{SRW}}_n(i) = y\right) P_y \left(J_n(T^\circ_{n,k+2-i}) = z\right). \tag{6.74}$$

Consider now the last factor in (6.74). By construction, $y \in \mathcal{V}^\star_n$. Hence, by (3.3),

$$P_y \left(J_n(T^\circ_{n,k+2-i}) = z\right) = \textstyle\sum_x \mathcal{L}^\star_{n,l}(y,x) P_x \left(J_n(T^\circ_{n,k+2-i}) = z\right). \tag{6.75}$$

where the sum is over x in $\partial\mathcal{V}^\star_n = \cup_{1\le l\le L^\star} \partial C^\star_{n,l}$ and where, in the notation of Lemma 6.7, $\mathcal{L}^\star_{n,l}(y,x) \equiv P_y\left(J_n(T^\star_{n,l}) = y\right)$, is the exit distribution from the set $C^\star_{n,l}$ containing y. Thus in particular, $\sum_{x\in\partial C^\star_{n,l}} \mathcal{L}^\star_{n,l}(y,x) = 1$. For indices i,k such that $k+2-i > 0$, we rewrite the probability in the remaining term as $P_x\left(J_n(T^\circ_{n,k+2-i}) = z\right) = J^>_{k+2-i}(x) + J^<_{k+2-i}(x)$ where, for $j \ge 1$,

$$J^>_j(x) \equiv P_x\left(J_n(T^\circ_{n,j}) = z, T^\star_n > j\right) \le P^{\mathsf{SRW}}_x\left(J^{\mathsf{SRW}}_n(j) = z\right), \tag{6.76}$$

$$J^<_j(x) \equiv P_x\left(J_n(T^\circ_{n,j}) = z, T^\star_n \le j\right). \tag{6.77}$$

(We reason as we did for I_1 to bound $J^>_j(x)$ in (6.76).) Consider first the contribution to I_2 coming from the terms $J^>_{k+2-i}(x)$, namely,

$$I^>_2 \equiv \sum_{k=1}^{\ell^\circ_n - 1} \sum_{i=1}^{k+2} \sum_{1\le l\le L^\star} \sum_{y\in C^\star_{n,l}} P^{\mathsf{SRW}}_z\left(T^{\mathsf{SRW},\star}_n = i, J^{\mathsf{SRW}}_n(i) = y\right) \sum_{x\in\partial C^\star_{n,l}} \mathcal{L}^\star_{n,l}(y,x) J^>_{k+2-i}(x).$$

To bound $I^>_2$ we relax the sum over i and use the bound (6.76) to write

$$I^>_2 \le \textstyle\sum_{i=1}^{\ell^\circ_n} \sum_{1\le l\le L^\star} \sum_{y\in C^\star_{n,l}} P^{\mathsf{SRW}}_z\left(T^{\mathsf{SRW},\star}_n = i, J^{\mathsf{SRW}}_n(i) = y\right) R_{n,l}(y) \tag{6.78}$$

where

$$R_{n,l}(y) = \textstyle\sum_{x\in\partial C^\star_{n,l}} \mathcal{L}^\star_{n,l}(y,x) \sum_{k=1}^{\ell^\circ_n - 1} P^{\mathsf{SRW}}_x\left(J^{\mathsf{SRW}}_n(k) = z\right). \tag{6.79}$$

We now split the sum over x in (6.79) according to whether $\mathrm{dist}(x,z) = 1$, $\mathrm{dist}(x,z) = 2$, or $\mathrm{dist}(x,z) \ge 3$ and use (6.52) of Lemma 6.14 to bound the sum over k: this gives

$$R_{n,l} \le c' \textstyle\sum_{d=1}^2 n^{-d} \sum_{x\in\partial C^\star_{n,l}:\mathrm{dist}(x,z)=d} \mathcal{L}^\star_{n,l}(y,x) + c'n^{-3} \tag{6.80}$$

where, by Lemma 6.7 and (6.28), $\mathcal{L}^\star_{n,l}(y,x) = (1+o(1))\mathcal{L}^\star_{n,l}(x) \le (1+o(1))n^{-1}$ for all $x \in \partial C^\star_{n,l}$. Hence, inserting (6.80) in (6.78),

$$I^>_2 \le c'n^{-3} + c_1 \max_{1\le l\le L^\star} P^{\mathsf{SRW}}_z\left(T^{\mathsf{SRW}}(C^\star_{n,l}) \le T^\star_n \le \ell^\circ_n\right) \sum_{d=1}^2 \sum_{1\le l\le L^\star} \frac{|\partial_d z \cap \partial C^\star_{n,l}|}{n^{d+1}}. \tag{6.81}$$

Throughout this proof $0 < c_i < \infty$, $i = 1, 2, \dots$ are constants. Now by (6.3), on $\mathcal{W}_n$, $\sum_{1\le l\le L^\star} |\partial_1 z \cap \partial C^\star_{n,l}| \le \kappa_\star$ and $\sum_{1\le l\le L^\star} |\partial_2 z \cap \partial C^\star_{n,l}| \le n/\log n$. Thus

$$I^>_2 \le c'n^{-3} + c_3 n^{-2} \max_{1\le l\le L^\star} P^{\mathsf{SRW}}_z\left(T^{\mathsf{SRW}}(C^\star_{n,l}) \le T^\star_n \le \ell^\circ_n\right). \tag{6.82}$$

To bound the last probability we write

$$P^{\mathsf{SRW}}_z\left(T^{\mathsf{SRW}}(C^\star_{n,l}) \le T^\star_n \le \ell^\circ_n\right) \le \textstyle\sum_{y\in C^\star_{n,l}} \sum_{k=1}^{\ell^\circ_n - 1} P^{\mathsf{SRW}}_z\left(J^{\mathsf{SRW}}_n(k) = y\right) \tag{6.83}$$

and split the sum over y according to whether $\mathrm{dist}(y,z) = 2$ or $\mathrm{dist}(y,z) \ge 3$. Using again (6.52) of Lemma 6.14, we then get

$$\textstyle\sum_{y\in C^\star_{n,l}:\mathrm{dist}(y,z)=2} P^{\mathsf{SRW}}_z\left(T^{\mathsf{SRW}}(y) \le T^\star_n \le \ell^\circ_n\right) \le \max_l |\partial_2 z \cap C^\star_{n,l}| n^{-2}, \tag{6.84}$$

$$\sum_{d\geq 3}\sum_{y\in C^\star_{n,l}:\mathrm{dist}(y,z)=d} P^{\mathsf{SRW}}_z\left(T^{\mathsf{SRW}}(y)\leq T^\star_n\leq \ell^\circ_n\right)\leq \max_l |C^\star_{n,l}|n^{-3}, \tag{6.85}$$

Now $\max_l|\partial_2 z\cap C^\star_{n,l}|\leq \max_l|C^\star_{n,l}|\leq c_4 n/\log n$ where the last inequality, valid on $\Omega^\star$, is (2.9) of Lemma 2.2. In view of this, plugging (6.84) and (6.85) in (6.83) and combining the result with (6.82), we finally get

$$I_2^>\leq c'n^{-3}+c_4(n^3\log n)^{-1}. \tag{6.86}$$

We now turn to the contribution to I_2 coming from the term $J^<_{k+2-i}(x)$. Since $J^<_{k+2-i}(x)$ is of the same nature as the probability appearing in (6.69), the straightforward idea is to iterate the decomposition (6.73)-(6.74). Doing so (6.67) becomes, for all $m\geq 3$,

$$\sum_{k=1}^{\ell^\circ_n-1} P^\circ_z\left(J^\circ_n(k+2)=z\right)=I_1+I_2^<+\cdots+I_m^<+I_m^>, \tag{6.87}$$

where, setting $q_i(x,y)\equiv P^{\mathsf{SRW}}_x\left(T^{\mathsf{SRW},\star}_n=i,J^{\mathsf{SRW}}_n(i)=y\right)$,

$$\begin{aligned} I_m^{\lessgtr} \quad \equiv \quad & \sum_{k=1}^{\ell^\circ_n-1}\sum_{i_1=1}^{k+2}\sum_{l_1}\sum_{y_1} q_{i_1}(z,y_1)\sum_{x_1}\mathcal{L}^\star_{n,l_1}(y_1,x_1)\dots\\ \dots \quad & \sum_{i_m=1}^{k+2-i_1\cdots-i_{m-1}}\sum_{l_m}\sum_{y_m} q_{i_m}(x_{m-1},y_m)\sum_{x_m}\mathcal{L}^\star_{n,l_m}(y_m,x_m)J^{\lessgtr}_{k+2-i_1\cdots-i_m}(x_m), \end{aligned}$$

for $J^>_j(x)$, $J^<_j(x)$ as in (6.76), (6.77), and with the convention that empty sums are zero. To bound $I_m^>$ we proceed as for $I_2^>$. More precisely, relaxing all sums over i_j and pushing the sum over k to $J^>_{k+2-i_1\cdots-i_m}(x_m)$ the last term in the resulting bound is

$$\sum_{i_m=1}^{\ell^\circ_n-1}\sum_{l_m}\sum_{y_m} q_{i_m}(x_{m-1},y_m)\sum_{k=1}^{\ell^\circ_n-1}\sum_{x_m}\mathcal{L}^\star_{n,l_m}(y_m,x_m)J^>_{k+2-i_1\cdots-i_m}(x_m), \tag{6.88}$$

which is of the same form as the r.h.s. of (6.78), and is bounded in the same way, the only difference being that the initial condition z of the probability law appearing in (6.82) now becomes x_{m-1}. But unlike z, which is at distance at least two from $C^\star_{n,l}$, x_{m-1} may be at distance one only. Thus, the leading contribution to the r.h.s. of (6.83) now is

$$\sum_{y\in C^\star_{n,l}:\mathrm{dist}(y,x_{m-1})=1} P^{\mathsf{SRW}}_{x_{m-1}}\left(T^{\mathsf{SRW}}(y)\leq T^\star_n\leq \ell^\circ_n\right)\leq \max_l|\partial_{x_{m-1}}\cap C^\star_{n,l}|n^{-1}, \tag{6.89}$$

where again $\max_l|\partial_1 z\cap C^\star_{n,l}|\leq \max_l|C^\star_{n,l}|\leq c_4 n/\log n$, and so,

$$I_m^<\leq c_5(n^2\log n)^{-1}. \tag{6.90}$$

We now turn to $I_m^>$. Here we use that for large enough m the chain will typically not revisit $\mathcal{V}^\star_n$ m times before time ℓ°_n. For this choose $m=n^2$. Set $\epsilon_n=1/\log n$ and $I_n=[\epsilon_n n^{2c_\star},\epsilon_n^{-1}n^{2c_\star}]$. By Lemma 6.12, $\max_{x\in\mathcal{V}_n}P^{\mathsf{SRW}}_x\left(T^{\mathsf{SRW}}(\mathcal{V}^\star_n\setminus x)\notin I_n\right)\leq c_6\epsilon_n$ on Ω^{SRW}, for large enough n. Now, at least $m'=m-\ell^\circ_n/(\epsilon_n n^{2c_\star})$ indices $i_1,\dots,i_m$ in $I_m^>$ must be smaller then $\epsilon_n n^{2c_\star}$. This readily yields, using the rough bound $J^>_{k+2-i_1\cdots-i_m}(x_m)\leq 1$, that $I_m^>\leq c_7\ell^\circ_n\epsilon_n^{m'}\binom{m}{m'}\ll I_m^<$. Plugging this and (6.90) in (6.88) with $m=n^2$, we get that

$$\sum_{k=1}^{\ell^\circ_n-1} P^\circ_z\left(J^\circ_n(k+2)=z\right)\leq c_8(\log n)^{-1}, \tag{6.91}$$

which is valid on $\Omega^{\mathsf{SRW}}\cap\Omega^\star$ for all but a finite number of indices n. The proof of assertion (i) of Proposition 6.2 is complete. □

Proof of Proposition 6.2, (ii). The proof is a rerun of the proof of assertion (i). We now briefly indicate the main modifications. Let $1\leq l'\leq L^\star$ and $z,z'\in\partial C^\star_{n,l'}$ be given, and assume first that $|z'\cap(\cup_{1\leq l\neq l'\leq L^\star}\partial C^\star_{n,l})|=0$, that is, z' lies in the boundary of a unique component $C^\star_{n,l'}$. As in (6.67) wecdecompose the probability in (6.5) into I_1+I_2 where I_1 and I_2 are the analogues of (6.68) and (6.69), respectively. Arguing as in (6.70)-(6.71) to bound I_1, but using (6.51) of Lemma 6.13 if $z=z'$ and (6.52) of Lemma 6.14 if $z\neq z'$,

$$I_1\leq \sum_{k=1}^{\ell^\circ_n-1} P^{\mathsf{SRW}}_z\left(J^{\mathsf{SRW}}_n(k)=z'\right)\leq c_1 n^{-1} \tag{6.92}$$

for some constant $0 < c_1 < \infty$. Turning to I_2 we write $I_2 = I_2^{>} + I_2^{<}$ as in the proof of assertion (i). To deal with $I_2^{>}$ we further distinguish two cases: (a) the chain visits $\mathcal{V}_n^\star \backslash C_{n,l'}^\star$ before visiting $C_{n,l'}^\star$ or (b) the converse occurs. The assumption that $|\partial z' \cap \partial \mathcal{V}_n^\star| \leq \kappa_\star$ guarantees that in case (a) the contribution to $I_2^{>}$ is at most $\mathcal{O}(1/(n^2 \log n))$, just as in (6.90) of assertion (ii). In case (b), the contribution to $I_2^{>}$ is bounded above by the sum of $p_n^\circ(z, z')$ (this corresponds to $k = 1$) and

$$\textstyle\sum_{k=2}^{\ell_n^\circ-1} \sum_{i=1}^{k} \sum_{y\in C_{n,l'}^\star} P_z^{\mathsf{SRW}}\left(T_n^{\mathsf{SRW},\star} = i, J_n^{\mathsf{SRW}}(i) = y\right) \sum_{x\in\partial C_{n,l'}^\star \backslash z'} \mathcal{L}_{n,l}^\star(y, x) J_{k-i}^{>}(x),$$

where $J_{k-i}^{>}(x)$ is defined as in (6.76) with z' substituted for z. Observing that each trajectory in the above quantity contains exactly one transition of the form $p_n^\circ(z_1, z_2)$, $z_1, z_2 \in \partial C_{n,l'}^\star$, we readily get that this term is at most

$$n^{-1}|\partial C_{n,l'}^\star \cap \partial z'| \max p_n^\circ(z_1, z_2) \leq \kappa_\star \max p_n^\circ(z_1, z_2) \leq \kappa_\star |C_{n,l'}^\star|/n^3 = \mathcal{O}(1/(n^2 \log n)),$$

where the last equality, valid on $\Omega^\star$, follows from the bound $p_n^\circ(z_1, z_2) \leq |C_{n,l'}^\star|/n^2$ together with (2.9) of Lemma 2.2. Iterating $m = n^2$ times as in (6.87), the sums of the contributions coming from case (b) is or order

$$\mathcal{O}(1/\log n) + \textstyle\sum_{x_1,x_2,\dots x_{m-1}\in\partial C_{n,l'}^\star \backslash z'} p_n^\circ(z, x_1) p_n^\circ(x_1, x_2) \dots p_n^\circ(x_{m-1}, z'), \tag{6.93}$$

where the sum is bounded above by $\max_{x_{m-1}} p_n^\circ(x_{m-1}, z') \leq |C_{n,l'}^\star|/n^2 = \mathcal{O}(1/n \log n)$ on $\Omega^\star$. Poceeding from there on as in the proof of assertion (i) readily yields the claim (6.5) of assertion (ii). The case where z' belongs to the boundary of several sets $C_{n,l}^\star$'s is a little more involved but goes along the same lines. We skip the details. This concludes the proof of Proposition 6.2. □

6.4. **Hitting time at stationarity: proof of Proposition 6.4.** Consider the continuous time Markov chain $(J_n^*(t), t > 0)$ with jump chain $(J_n^\circ(k), k \in \mathbb{N})$ and rate one exponential waiting times. That is, given a family $(e_{n,i}^*, i \in \mathbb{N})$ of independent mean one exponential r.v.'s, independent of J_n°,

$$J_n^*(t) = J_n^\circ(i) \;\text{ if }\; s_n(i) \leq t < s_n(i+1) \;\text{ for some } i, \tag{6.94}$$

where $s_n(k) \equiv \sum_{i=0}^{k-1} e_{n,i}^*$, $k \in \mathbb{N}$. Write P_x^* for the law of J_n^* started in x. Let us first prove, that under the assumptions of Proposition 6.4, (6.10) holds for the continous time Markov chain J_n^*. For this we use results from [1]. Set $B \equiv A \cap I_n^\star$ and write $T^*(B) \equiv \inf\{t > 0 \mid J_n^*(t) \in B\}$. Then, by Theorem 3 and Lemma 2 of [1] we have,

$$P_{\pi_n^\circ}^*\left(T^*(B) > t\right) \geq \left(1 - \tau_n^\circ \tfrac{q(B,B^c)}{1-\pi_n^\circ(B)}\right) \exp\left(-t \tfrac{q(B,B^c)}{1-\pi_n^\circ(B)}\right), \quad t > 0, \tag{6.95}$$

where τ_n° is as in (6.30) and where $q(B, B^c) = \sum_{x\in B} \sum_{y\notin B} \pi_n^\circ(x) p_n^\circ(x, y) = \pi_n^\circ(B)$ as follows from (6.12) and the fact that $B \subseteq I_n^\star$. By Proposition 6.3,

$$\pi_n^\circ(B) = |B|/|\mathcal{V}_n^\circ| \leq |I_n^\star|/|\mathcal{V}_n^\circ| \leq n^{-c_\star}(1 + o(1)), \tag{6.96}$$

where we used (6.7) and (2.10) in the last inequality. From this and (6.30), we get that

$$P_{\pi_n^\circ}^*\left(T^*(B) > t\right) \geq \left(1 - n^{-(c_\star-2)}(1 + o(1))\right) \exp\left(-t \tfrac{|B|}{|\mathcal{V}_n^\circ|}(1 + o(n^{-c_\star}))\right), \quad t > 0. \tag{6.97}$$

The idea then is that for s_n as in (6.94), $T^*(B) - T^\circ(B) = s_n(T^\circ(B)) - T^\circ(B)$, which should be small for $T^\circ(B)$ large. Indeed, a classical large deviation estimates yields that if $0 < m_n \uparrow \infty$ is an integer valued sequence then for all $\zeta > 0$

$$P_x^*\left(|s_n(m_n) - m_n| \geq \zeta m_n\right) \leq 2e^{-m_n\{\zeta - \log(1+\zeta)\}}. \tag{6.98}$$

We thus need an a priori lower bound on $T^{\circ}(B)$. To this end note that $B \subset V_n(\rho_n^\star)$ so that by Theorem 1.1 of [16], for $\rho_n^\star$ as in (2.3) and any $c_\star$ such that $n^{c_\star} \gg n\log n$, we have for all $l_n \leq n^{c_\star}/\log n$ that

$$P^{\circ}_{\pi_n^\circ}(T^\circ(B) > l_n) \geq (1-\pi_n^\circ(B)) \inf_{x\notin B} P_x^{\mathsf{SRW}}(T^{\mathsf{SRW}}(V_n(\rho_n^\star)\setminus x) \geq l_n) \geq 1 - \mathcal{O}(\tfrac{1}{\log n}).$$

where we used (6.96) in the last inequality. From this bound, (6.98), and (6.97) we get that for any $\zeta > 0$,

$$P^{*}_{\pi_n^\circ}(T^*(B) > t) \leq P^{\circ}_{\pi_n^\circ}(T^\circ(B) > t/(1+\zeta)) + 2e^{-l_n\{\zeta - \log(1+\zeta)\}} + \mathcal{O}(\tfrac{1}{\log n}). \quad (6.99)$$

Taking e.g. $\zeta = 1/2$ and $l_n = n^{c_\star/2}$ yields (6.10) and finishes the proof of Proposition 6.4.

7. Proof of Theorem 3.3 and of Theorem 3.1

The proofs of Theorem 3.3 and of Theorem 3.1 hinge upon the next two lemmata.

7.1. **Preparatory Lemmata.** Let $0 < \rho < 1$ and, for $V_n(\rho)$ defined in (2.1), set

$$C^\star_{n,l}(\rho) = \begin{cases} C^\star_{n,l} & \text{if } C^\star_{n,l} \cap V_n(\rho) \neq \emptyset, \\ \emptyset & \text{else.} \end{cases} \quad (7.1)$$

Lemma 7.1. *Assume that $c_\star > 2$. There exists a subset $\Omega_3 \subset \Omega$ with $\mathbb{P}(\Omega_3) = 1$ such that on Ω_3, for all but a finite number of indices n, for all $\rho_n^\star \leq \rho \leq 1 - 3\rho_n^\star$,*

$$\left|\cup_{1\leq l\leq L^\star} C^\star_{n,l}(\rho)\right| / |\mathcal{V}_n^\circ| \leq n^{-c_\star+1} 2^{-n\rho}(1+o(1)), \quad (7.2)$$

and, for $m^\star_{n,l}(x)$ as in (6.14),

$$\sum_{1\leq l\leq L^\star} \sum_{x\in\partial C^\star_{n,l}(\rho)} \pi_n^\circ(x) m^\star_{n,l}(x) \leq n^{-c_\star+1} 2^{-n\rho}(1+o(1)). \quad (7.3)$$

Lemma 7.2. *Assume that $c_\star > 2$. On $\Omega^\star$, for all but a finite number of indices n,*

$$\pi_n^\circ\left(\partial\left(\cup_{1\leq l\leq L^\star} C^\star_{n,l}\right)\right) \leq n^{-2(c_\star-1)}(1+\mathcal{O}(n^{-(c_\star-1)})). \quad (7.4)$$

Proof of Lemma 7.2. By (6.6), $\pi_n^\circ\left(\partial(\cup_{1\leq l\leq L^\star} C^\star_{n,l})\right) \leq n|\cup_{1\leq l\leq L^\star} C^\star_{n,l}|/|\mathcal{V}_n^\circ|$. By (2.12) of Lemma 2.2 and (6.7) of Proposition 6.3, on $\Omega^\star$, for all but a finite number of indices n,

$$n|\cup_{1\leq l\leq L^\star} C^\star_{n,l}|/|\mathcal{V}_n^\circ| = n\left|V_n(\rho_n^\star)\setminus I_n^\star\right|/|\mathcal{V}_n^\circ| \leq n n^{-2c_\star+1}(1+\mathcal{O}(n^{-(c_\star-1)})), \quad (7.5)$$

proving (7.4). ∎

Proof of Lemma 7.1. Set $k_n^\star \equiv \max_{2\leq l\leq L^\star} |C^\star_{n,l}(\rho)|$ and let

$$S_n(k) \equiv \textstyle\sum_{l=2}^{L^\star} |C^\star_{n,l}(\rho)| \mathbb{1}_{\{|C^\star_{n,l}(\rho)|=k\}} \quad (7.6)$$

be the total number of vertices that belong to sets $C^\star_{n,l}(\rho)$ that have cardinality k. Note that by (2.3) and (2.9) of Lemma 2.2, on $\Omega^\star$, for large enough n,

$$k_n^\star \leq n/((c_\star - 2)\log n). \quad (7.7)$$

Now, on the one hand,

$$|\cup_{1\leq l\leq L^\star} C^\star_{n,l}(\rho)|/|\mathcal{V}_n^\circ| = \tfrac{1}{|\mathcal{V}_n^\circ|} \textstyle\sum_{l=2}^{L^\star} |C^\star_{n,l}(\rho)| = \tfrac{1}{|\mathcal{V}_n^\circ|} \sum_{k=2}^{k_n^\star} S_n(k). \quad (7.8)$$

On the other hand, by (6.6)

$$\textstyle\sum_{1\leq l\leq L^\star} \sum_{x\in\partial C^\star_{n,l}(\rho)} \pi_n^\circ(x) m^\star_{n,l}(x) \leq \tfrac{1}{|\mathcal{V}_n^\circ|} \sum_{k=2}^{k_n^\star} S_n(k), \quad (7.9)$$

where we used in the final inequality that by (6.14),

$$\textstyle\sum_{x\in\partial C^{\star}_{n,l}(\rho)} m^{\star}_{n,l}(x) \;=\; n^{-1}\sum_{y\in C^{\star}_{n,l}(\rho)} |\partial C^{\star}_{n,l}(\rho)\cap\partial y| \le |C^{\star}_{n,l}(\rho)| \tag{7.10}$$

since $|\partial C^{\star}_{n,l}(\rho)\cap\partial y| \le n$. Let us now focus on the quantities $S_n(k)$, $2\le k\le k^{\star}_n$. We claim that if $c_\star > 2$ there exists a subset $\Omega^{\star\star}\subset\Omega$ with $\mathbb{P}(\Omega^{\star\star})=1$ such that, on $\Omega^{\star\star}$, for all but a finite number of indices n, for all $\rho^{\star}_n\le\rho\le 1-3\rho^{\star}_n$,

$$S_n(2) \le n^{-c_\star+1}2^{n(1-\rho)}(1+\mathcal{O}(n^{-(c_\star-1)})), \tag{7.11}$$

$$S_n(3) \le n^{-2(c_\star-1)}2^{n(1-\rho)}(1+\mathcal{O}(n^{-(c_\star-1)})), \tag{7.12}$$

and, for all $4\le k\le k^{\star}_n$,

$$S_n(k) \le n^{-1}n^{-c_\star+1}2^{n(1-\rho)}(1+\mathcal{O}(n^{-(c_\star-1)})). \tag{7.13}$$

We first prove (7.11). For this let us introduce the variables $\chi^{\rho}(x)\equiv \mathbb{1}_{\{w_n(x)\ge r_n(\rho)\}}$, $\chi^{\star,\rho}_n(x)\equiv\mathbb{1}_{\{r_n(\rho^{\star}_n)\le w_n(x)<r_n(\rho)\}}$, and $\chi_n(x)\equiv\mathbb{1}_{\{w_n(x)\ge r_n(\rho^{\star}_n)\}}$. They are Bernoulli r.v.'s with $\mathbb{P}(\chi^{\rho}(x)=1)=2^{-\rho n}$, $\mathbb{P}(\chi_n(x)=1)=n^{-c_\star}$, and $\mathbb{P}(\chi^{\star,\rho}_n(x)=1)=n^{-c_\star}-2^{-\rho n}$ respectively, that inherit the independence of the variables $(w_n(x), x\in\mathcal{V}_n)$. We then may write $S_n(2)=S^0_n(2)+S^1_n(2)$ where, for $\mathcal{G}_2$ defined below (6.40),

$$S^0_n(2) \;\equiv\; \textstyle\sum_{\mathcal{C}=\{x,y\}\in\mathcal{G}_2}(Y_n(x,y)+Y_n(y,x)), \tag{7.14}$$

$$S^1_n(2) \;\equiv\; \textstyle\sum_{\mathcal{C}=\{x,y\}\in\mathcal{G}_2} Z_n(x,y), \tag{7.15}$$

and

$$Y_n(x,y) \;\equiv\; \chi^{\rho}_n(x)\chi^{\star,\rho}_n(y)\textstyle\prod_{z\in(\partial x\cup\partial y)\setminus\{x,y\}}(1-\chi_n(z)), \tag{7.16}$$

$$Z_n(x,y) \;\equiv\; \chi^{\rho}_n(x)\chi^{\rho}_n(y)\textstyle\prod_{z\in(\partial x\cup\partial y)\setminus\{x,y\}}(1-\chi_n(z)). \tag{7.17}$$

To bound $S^0_n(2)$ and $S^1_n(2)$ we proceed as in the proof of Lemma 6.10, i.e. we decompose $\mathcal{G}_2$ into $\mathcal{G}_2=\cup_{1\le j\le n}\cup_{1\le i\le v_n}\mathcal{G}^{j,i}_2$, $v_n<2n^4$, where the $\mathcal{G}^{j,i}_2$'s are defined in (6.42), and use a concentration bound to estimate the sum (of now independent r.v.'s) over each $\mathcal{G}^{j,i}_2$. Doing this we readily get that $\mathbb{E}S^0_n(2)=n(n^{-c_\star}-2^{-\rho n})2^{n(1-\rho)}(1-n^{-c_\star})^{2(n-1)}$ and

$$\mathbb{P}\left(|S^0_n(2)-\mathbb{E}S^0_n(2)|\ge 2n\sqrt{\mathbb{E}S^0_n(2)}\right)\le nv_ne^{-n}. \tag{7.18}$$

Similarly, $\mathbb{E}S^1_n(2)=n2^{n(1-2\rho)}(1-n^{-c_\star})^{2(n-1)}$ and for all $\rho^{\star}_n\le\rho\le(1-4\rho^{\star}_n)/2$,

$$\mathbb{P}\left(|S^1_n(2)-\mathbb{E}S^1_n(2)|\ge 2n\sqrt{\mathbb{E}S^1_n(2)}\right)\le nv_ne^{-n}. \tag{7.19}$$

For $\rho>(1-4\rho^{\star}_n)/2$ we simply use that by Tchebychev's first order order inequality,

$$\mathbb{P}\left(S^1_n(2)\ge 2^{-n\rho/2}\mathbb{E}S^0_n(2)\right)\le 2^{-n\rho/2}. \tag{7.20}$$

From the assumptions that $\rho^{\star}_n\le\rho\le 1-3\rho^{\star}_n$ and $c_\star>1$ it then immediately follows that (7.11) holds true with a probability larger than $1-c_0n^5e^{-c_1n}$ for some constants $0<c_0,c_1<\infty$. Thus, by Borel-Cantelli Lemma, it holds on a subset of Ω of full measure, for all but a finite number of indices n. One proves (7.12) in a similar way.

When $4\le k\le k^{\star}_n$ we do not need such a refined control on $S_n(k)$: we simply write

$$S_n(k)\le k\textstyle\sum_{B\subset\mathcal{V}_n}\mathbb{1}_{\{\exists x\in B:\chi^{\rho}_n(x)\prod_{y\in B\setminus x}\chi^{\star,\rho}_n(y)=1\}}\prod_{z\in\partial B}(1-\chi_n(z)), \tag{7.21}$$

where the sum is over all subsets $B\subset\mathcal{V}_n$ such that $|B|=k$, and such that the graph $G(B)$ is connected. Since the number of such sets is bounded above by $(k-1)!n^{k-1}2^n$, $\mathbb{E}S_n(k)\le k!n^{k-1}n^{-c_\star(k-1)}2^{n(1-\rho)}$, and a first order order Tchebychev inequality yields

$$\mathbb{P}\left(S_n(k)\ge n^{-1}\mathbb{E}S^0_n(2)\right)\le k!nn^{-(c_\star-1)(k-2)}. \tag{7.22}$$

One easily checks that if $c_\star > 2$ then, for all $m \geq 3$ and all $K \leq n$,

$$\sum_{k=m}^{K} k! n^{-(c_\star-1)(k-2)} \leq (m!+1) n^{-(m-2)(c_\star-1)}. \tag{7.23}$$

Therefore $\mathbb{P}\left(\cup_{4\leq k\leq k_n^\star}\left\{S_n(k)\geq n^{-1}\mathbb{E}S_n^0(2)\right\}\right)\leq 25n^{-2(c_\star-1)+1}$, which is summable when $c_\star > 2$. By Borel-Cantelli Lemma we conclude that on a subset of Ω of full measure, for all but a finite number of indices n, (7.13) holds true for all $4 \leq k \leq k_n^\star$. This concludes the proof of the claim (7.11)-(7.13).

Now, by (7.11)-(7.13) and (6.7), on $\Omega_3 \equiv \Omega^\star \cap \Omega^{\star\star}$, for all large enough n,

$$\frac{1}{|\mathcal{V}_n^\circ|}\sum_{k=2}^{k_n^\star} S_n(k) \leq (1+o(1))(1+(k_n^\star/n))n^{-c_\star+1}2^{-n\rho} = n^{-c_\star+1}2^{-n\rho}(1+o(1)), \tag{7.24}$$

where the last equality follows from (7.7). Inserting (7.24) in (7.8) and in (7.9) yields (7.2) and (7.3), respectively. The proof of Lemma 7.1 is done. □

7.2. **Elementary properties of the chains $J_n^\dagger$ and J_n°.** For easy reference we gather here a few elementary properties of the chains $J_n^\dagger$ and J_n°. We state them without proof: recalling that $J_n^\circ(i) \equiv J_n(T_{n,i}^\circ)$ and $J_n^\dagger(i) \equiv J_n(T_{n,i}^\dagger)$ they are immediate consequences from the definitions of the sequences $\left(T_{n,j}^\dagger\right)$ and $\left(T_{n,j}^\circ\right)$ (see (3.3)-(3.10)).

Lemma 7.3. *To each $j \geq 0$ there corresponds a unique $i \leq j$ such that:*
(i) $J_n^\dagger(j) \notin \mathcal{V}_n^\circ \Leftrightarrow T_{n,j-1}^\dagger = T_{n,i-1}^\circ < T_{n,j}^\dagger = T_{n,i-1}^\circ + 1 < T_{n,j+1}^\dagger = T_{n,i}^\circ$,
(ii) $J_n^\dagger(j) \in \mathcal{V}_n^\circ \Leftrightarrow T_{n,j}^\dagger = T_{n,i}^\circ$.

From Lemma 7.3, (i), we derive two descriptions of the event $\{J_n^\dagger(j) \in C_{n,l}^\star\}$, $j > 0$, $1 \leq l \leq L^\star$. The first consists in saying that a visit of $J_n^\dagger$ to $C_{n,l}^\star$ must be immediately preceded and followed by a visit to $\partial C_{n,l}^\star$.

Corollary 7.4. $\{J_n^\dagger(j) \in C_{n,l}^\star\} = \{J_n^\dagger(j-1) \in \partial C_{n,l}^\star, J_n^\dagger(j) \in C_{n,l}^\star, J_n^\dagger(j+1) \in \partial C_{n,l}^\star\}$.

The second expresses the fact that when $J_n^\dagger(j)$ enters $C_{n,l}^\star$, $J_n^\circ(i)$ straddles over it.

Corollary 7.5. *To each $j \geq 0$ there corresponds a unique $i \leq j$ such that $T_{n,j}^\dagger = T_{n,i-1}^\circ + 1$, and so*

$$\{J_n(T_{n,j}^\dagger) \in C_{n,l}^\star\} = \{J_n(T_{n,i-1}^\circ) \in \partial C_{n,l}^\star, J_n(T_{n,i}^\circ) \in \partial C_{n,l}^\star\}. \tag{7.25}$$

Note finally that by Lemma 7.3, (ii), the chain $J_n^\dagger$ observed only when it visits $\mathcal{V}_n^\circ$ is nothing but the chain J_n° itself:

Corollary 7.6. $(J_n^\dagger(j) : \exists i > 0 \text{ } s.t. \text{ } T_{n,j}^\dagger = T_{n,i}^\circ, j \geq 0) \stackrel{d}{=} (J_n^\circ(i), i \geq 0)$.

7.3. **Proof of Theorem 3.1.** Theorem 3.1 is a rough estimate. By (3.15),

$$0 \leq k_n^\dagger(t) - k_n^\circ(t) = \sum_{j=0}^{k_n^\dagger(t)-1} \mathbb{1}_{\{J_n^\dagger(j)\notin\mathcal{V}_n^\circ\}}. \tag{7.26}$$

We now want to replace the chain $J_n^\dagger$ and the quantity $k_n^\dagger(t)$ in the right hand side of (7.26) by, respectively, J_n° and $k_n^\circ(t)$. Note that by Corollary 7.4, for each $j \geq 1$,

$$\{J_n^\dagger(j) \notin \mathcal{V}_n^\circ\} = \cup_{1\leq l\leq L^\star}\{J_n^\dagger(j) \in C_{n,l}^\star\} \subseteq \{J_n^\dagger(j-1) \in \partial(\cup_{1\leq l\leq L^\star} C_{n,l}^\star)\}. \tag{7.27}$$

From this and the fact that $J_n^\dagger(0) = J_n^\circ(0) \in \mathcal{V}_n^\circ$ (indeed $J_n^\dagger$ starts in π_n°), we deduce that,

$$\begin{aligned}\sum_{j=0}^{k_n^\dagger(t)-1} \mathbb{1}_{\{J_n^\dagger(j)\notin\mathcal{V}_n^\circ\}} &\leq \sum_{j=1}^{k_n^\dagger(t)-1} \mathbb{1}_{\{J_n^\dagger(j-1)\in\partial(\cup_{1\leq l\leq L^\star} C_{n,l}^\star)\}} &(7.28)\\ &\stackrel{d}{=} \sum_{i=1}^{k_n^\circ(t)-1} \mathbb{1}_{\{J_n^\circ(i-1)\in\partial(\cup_{1\leq l\leq L^\star} C_{n,l}^\star)\}}, &(7.29)\end{aligned}$$

where the last equality follows from Corollary 7.6 and the definition of $k_n^\circ(t)$ (see (3.19)). It remains to bound the last sum in (7.29). Since $k_n^\circ(t) = \lfloor a_n t \rfloor$ is deterministic, a first order Tchebychev inequality entails that for all $c_\circ > 0$,

$$P_{\pi_n^\circ}\left(\sum_{i=1}^{\lfloor a_n t\rfloor -1} \mathbb{1}_{\{J_n^\circ(i-1)\in\partial(\cup_{1\le l\le L^\star} C_{n,l}^\star)\}} \ge n^{-c_\circ}\lfloor a_n t\rfloor\right) \le n^{c_\circ}\pi_n^\circ\left(\partial(\cup_{1\le l\le L^\star} C_{n,l}^\star)\right).$$

Inserting (7.4) of Lemma 7.2 in the right hand side above, and combining the resulting bound with (7.26) and (7.29), we get that on $\Omega^\star$, for all but a finite number of indices n,

$$P_{\pi_n^\circ}\left(k_n^\dagger(t) \ge k_n^\circ(t)\left(1+n^{-c_\circ}\right)\right) \le n^{-2(c_\star-1)+c_\circ}(1+\mathcal{O}(n^{-(c_\star-1)})). \tag{7.30}$$

This readily implies the claim of Theorem 3.1.

7.4. **Proof of Theorem 3.3.** By definition of the Skorohod topology on $D[0,\infty)$, it is enough to show this result with ρ_∞ replaced by ρ_r, the Skorohod metric on $D[0,r]$, for $r > 0$ arbitrary. Choosing $r = 1$ for convenience we get

$$\mathcal{P}_{\pi_n^\circ}\left(\rho_1\big(S_n(\cdot), S_n^\circ(\cdot)\big) > n^{1-c_\star/2}\right) \le \mathcal{P}_{\pi_n^\circ}\big(\sup_{0\le t\le 1}\widehat{S}_n(t) > n^{1-c_\star/2}\big). \tag{7.31}$$

Theorem 3.3 then is an immediate consequence of the lemma below.

Lemma 7.7. *Assume that $c_\star > 2$ and that $\beta > \beta_c(\varepsilon)$. Then $\mathbb{P}$-almost surely,*

$$\limsup_{n\to\infty}\mathcal{P}_{\pi_n^\circ}\big(\sup_{0\le t\le 1}\widehat{S}_n(t) > n^{1-c_\star/2}\big) = 0. \tag{7.32}$$

Proof of Lemma 7.7. Since $\widehat{S}_n$ is nondecreasing,

$$\mathcal{P}_{\pi_n^\circ}\big(\sup_{0\le t\le 1}\widehat{S}_n(t) > \epsilon\big) \le \mathcal{P}_{\pi_n^\circ}\big(\widehat{S}_n(1) > \epsilon\big) \tag{7.33}$$

for all $\epsilon > 0$. Introducing the event

$$\mathcal{A} \equiv \left\{\forall_{0\le j\le k_n^\dagger(1)-1}\forall_{1\le l\le L^\star}\widehat{\Lambda}_n^\dagger(j)\mathbb{1}_{\{J_n^\dagger(j)\in C_{n,l}^\star\}} \le n2\bar{\varrho}_{n,l}(0)\right\} \tag{7.34}$$

we have by Corollary 4.2 that on $\Omega^\star$, for all but a finite number of indices n,

$$\mathcal{P}_{\pi_n^\circ}\big(\widehat{S}_n(1) > \epsilon\big) \le e^{-n} + n^{-2(c_\star-1)+c_\circ} + \mathcal{P}_{\pi_n^\circ}\big(\widehat{S}_n(1) > \epsilon, \mathcal{A}\big), \tag{7.35}$$

where $c_\circ > 0$ is arbitrary. From the definitions (3.27) and (3.12) of $\widehat{S}_n$ and $\Lambda_n^\dagger(i)$, and since $\Lambda_n^\dagger(i)$ is non zero if and only if $J_n^\dagger(i) \in \cup_{1\le l\le L^\star} C_{n,l}^\star$, we see that on $\mathcal{A}$,

$$\widehat{S}_n(1) = c_n^{-1}\sum_{l=1}^{L^\star}\sum_{j=1}^{k_n^\dagger(1)-1}\widehat{\Lambda}_n^\dagger(j)\mathbb{1}_{\{J_n^\dagger(j)\in C_{n,l}^\star\}} \tag{7.36}$$

$$\le 2c_n^{-1}\sum_{l=1}^{L^\star}\sum_{j=1}^{k_n^\dagger(1)-1} n\bar{\varrho}_{n,l}(0)\mathbb{1}_{\{J_n^\dagger(j)\in C_{n,l}^\star\}}. \tag{7.37}$$

Therefore,

$$\mathcal{P}_{\pi_n^\circ}\big(\widehat{S}_n(1) > \epsilon, \mathcal{A}\big) \le \mathcal{P}_{\pi_n^\circ}\big(2nc_n^{-1}\sum_{l=1}^{L^\star}\bar{\varrho}_{n,l}(0)\sum_{j=1}^{k_n^\dagger(1)-1}\mathbb{1}_{\{J_n^\dagger(j)\in C_{n,l}^\star\}} > \epsilon\big). \tag{7.38}$$

The problem we still face is that the quantity $\bar{\varrho}_{n,l}(0)$ appearing in (7.38) can be very large compared to c_n. However, sets $C_{n,l}^\star$ such that this happens will typically not be visited. More precisely, for $C_{n,l}^\star(\rho)$ as in (7.1), one may choose the parameter $0 < \rho < 1$ in a such a way that the event

$$\widetilde{\mathcal{A}} \equiv \left\{\forall_{1\le j\le k_n^\dagger(1)-1} J_n^\dagger(j) \notin \left(\cup_{1\le l\le L^\star} C_{n,l}^\star(\rho)\right)\right\}, \tag{7.39}$$

has probability close to one. Indeed

$$\mathcal{P}_{\pi_n^\circ}\big(\widetilde{\mathcal{A}}^c\big) = P_{\pi_n^\circ}\big(\widetilde{\mathcal{A}}^c\big) = \sum_{1\leq l\leq L^\star} E_{\pi_n^\circ} \sum_{j=0}^{k_n^\dagger(1)-1} \mathbb{1}_{\{J_n^\dagger(j)\in C_{n,l}^\star\}} \tag{7.40}$$

$$= \sum_{1\leq l\leq L^\star} E_{\pi_n^\circ} \sum_{i=1}^{k_n^\circ(1)-1} \mathbb{1}_{\{J_n^\circ(i-1)\in\partial C_{n,l}^\star, J_n^\circ(i)\in\partial C_{n,l}^\star\}} \tag{7.41}$$

where (7.41) follows from Corollary 7.5. Note that for all $x\in\partial C_{n,l}^\star$,

$$P_{\pi_n^\circ}\left(J_n^\circ(i-1)=x, J_n^\circ(i)\in\partial C_{n,l}^\star\right) = \pi_n^\circ(x)P_x(J_n^\circ(1)\in\partial C_{n,l}^\star) = \pi_n^\circ(x)m_{n,l}^\star(x), \tag{7.42}$$

where $m_{n,l}^\star(x)$ is defined in (6.14). Inserting this in (7.41), it follows from (7.3) of Lemma 7.1 that on Ω_3, for all but a finite number of indices n,

$$\mathcal{P}_{\pi_n^\circ}\big(\widetilde{\mathcal{A}}^c\big) \leq k_n^\circ(t)\sum_{1\leq l\leq L^\star}\sum_{x\in\partial C_{n,l}^\star(\rho)} \pi_n^\circ(x)m_{n,l}^\star(x) \tag{7.43}$$

$$\leq k_n^\circ(t) n^{-c_\star+1}2^{-n\rho}(1+o(1)) = n2^{-n\rho}2^{n\varepsilon_n - n\rho_n^\star}(1+o(1)). \tag{7.44}$$

where we wrote $\varepsilon_n \equiv \frac{\log a_n}{n\log 2}$; thus by 1.19, $\lim_{n\to\infty}\varepsilon_n=\varepsilon$, $0<\varepsilon<1$. Assume from now on that $\omega\in\Omega_3$ and take $\rho\equiv\varepsilon_n-\rho_n^\star/2$. Then

$$\mathcal{P}_{\pi_n^\circ}\big(\widehat{S}_n(1)>\epsilon,\mathcal{A}\big) \leq \mathcal{P}_{\pi_n^\circ}\big(\widetilde{\mathcal{A}}^c\big) + \mathcal{P}_{\pi_n^\circ}\big(\widehat{S}_n(1)>\epsilon,\mathcal{A},\widetilde{\mathcal{A}}\big)$$

$$\leq n2^{-n\rho_n^\star/2}(1+o(1)) + \mathcal{P}_{\pi_n^\circ}\big(\widehat{\mathcal{A}}\big), \tag{7.45}$$

where, recalling from (2.1) that $V_n(\varepsilon_n-\rho_n^\star/2)=\{x\in\mathcal{V}_n \mid w_n(x)\geq r_n(\varepsilon_n-\rho_n^\star/2)\}$,

$$\widehat{\mathcal{A}} \equiv \left\{2nc_n^{-1}\sum_{1\leq l\leq L^\star\,:\,C_{n,l}^\star\cap V_n(\varepsilon_n-\rho_n^\star/2)=\emptyset} \bar{\varrho}_{n,l}(0)\sum_{j=1}^{k_n^\dagger(1)-1}\mathbb{1}_{\{J_n^\dagger(j)\in C_{n,l}^\star\}} > \epsilon\right\}. \tag{7.46}$$

Again, we wish to express this event in terms of the chain J_n° and the quantity $k_n^\circ(t)$ rather than $J_n^\dagger$ and $k_n^\dagger(t)$. For this note that by Corollary 7.5, Corollary 7.6, the definition (3.15) of $k_n^\dagger(t)$ and the definition (3.19) of $k_n^\circ(t)$, for each $1\leq l\leq L^\star$,

$$\sum_{j=1}^{k_n^\dagger(1)-1}\mathbb{1}_{\{J_n^\dagger(j)\in C_{n,l}^\star\}} \stackrel{d}{=} \sum_{i=1}^{k_n^\circ(1)-1}\mathbb{1}_{\{J_n^\circ(i-1)\in\partial C_{n,l}^\star, J_n^\circ(i)\in\partial C_{n,l}^\star\}}. \tag{7.47}$$

Then, by Tchebychev inequality, (7.47), and (7.42),

$$\mathcal{P}_{\pi_n^\circ}\big(\widehat{\mathcal{A}}\big) \leq \frac{2n\lfloor a_n\rfloor}{\epsilon c_n}\sum_{\substack{1\leq l\leq L^\star:\\ C_{n,l}^\star\cap V_n(\varepsilon_n-\rho_n^\star/2)=\emptyset}} \max_{x\in C_{n,l}^\star} w_n(x)\sum_{x\in\partial C_{n,l}^\star}\pi_n^\circ(x)m_{n,l}^\star(x). \tag{7.48}$$

We next decompose the sum in (7.48) according to the size of $\max_{x\in C_{n,l}^\star} w_n(x)$: given $K>0$ to be chosen later define, for $0\leq k\leq K$,

$$\mathcal{I}_k \equiv \left\{1\leq l\leq L^\star \mid r_n\big(\varepsilon_n-\tfrac{k+2}{2}\rho_n^\star\big) \leq \max_{x\in C_{n,l}^\star} w_n(x) \leq r_n\big(\varepsilon_n-\tfrac{k+1}{2}\rho_n^\star\big)\right\}. \tag{7.49}$$

By this and the choices of a_n and c_n from Theoreom 1.1, (7.48) becomes

$$\mathcal{P}_{\pi_n^\circ}\big(\widehat{\mathcal{A}}\big) \leq 2n\epsilon^{-1}\left(\sum_{0\leq k\leq K} Q_{n,k} + R_n\right), \tag{7.50}$$

where

$$Q_{n,k} = 2^{n\varepsilon_n} r_n^{-1}(\varepsilon_n) r_n\big(\varepsilon_n-\tfrac{k+1}{2}\rho_n^\star\big)\sum_{l\in\mathcal{I}_k}\sum_{x\in\partial C_{n,l}^\star}\pi_n^\circ(x)m_{n,l}^\star(x), \tag{7.51}$$

$$R_n = 2^{n\varepsilon_n} r_n^{-1}(\varepsilon_n) r_n\big(\varepsilon_n-\tfrac{K+2}{2}\rho_n^\star\big)\sum_{1\leq l\leq L^\star}\sum_{x\in\partial C_{n,l}^\star}\pi_n^\circ(x)m_{n,l}^\star(x). \tag{7.52}$$

Now,

$$\sum_{l\in\mathcal{I}_k}\sum_{x\in\partial C_{n,l}^\star}\pi_n^\circ(x)m_{n,l}^\star(x) \leq \sum_{1\leq l\leq L^\star}\sum_{x\in\partial C_{n,l}^\star\left(\varepsilon_n-\frac{k+2}{2}\rho_n^\star\right)}\pi_n^\circ(x)m_{n,l}^\star(x)$$

$$\leq n^{-c_\star+1}2^{-n\left(\varepsilon_n-\frac{k+2}{2}\rho_n^\star\right)}(1+o(1)) \tag{7.53}$$

where the last inequality is (7.3) of Lemma 7.1. Inserting (7.53) in (7.51),

$$Q_{n,k} \le n2^{\frac{kn}{2}\rho_n^\star} r_n^{-1}(\varepsilon_n) r_n\big(\varepsilon_n - \tfrac{k+1}{2}\rho_n^\star\big). \tag{7.54}$$

Using (2.25), the bound $\sqrt{1-x}-1 \le -\frac{1}{2}x(1+\frac{1}{4}x)$, $0<x<1$, and the assumption that $\beta > \beta_c(\varepsilon)$, so that $\alpha(\varepsilon_n) \equiv \beta_c(\varepsilon_n)/\beta < 1$ for large enough n, it follows from (7.54) that

$$Q_{n,k} \le c_0 n 2^{-n\rho_n^\star/\alpha(\varepsilon_n)} 2^{-n\rho_n^\star(1/\alpha(\varepsilon_n)-1)\frac{k}{2}} \tag{7.55}$$

for some constant $0 < c_0 \equiv c_0(\varepsilon_n, \beta) < \infty$. Similarly, by (7.3) with $\rho = \rho_n^\star$,

$$\textstyle\sum_{1\le l\le L^\star} \sum_{x\in\partial C^\star_{n,l}} \pi_n^\circ(x) m^\star_{n,l}(x) \le n^{-2c_\star+1}(1+o(1)) \tag{7.56}$$

and

$$R_n \le n2^{n\varepsilon_n - 2n\rho_n^\star} r_n^{-1}(\varepsilon_n) r_n\big(\varepsilon_n - \tfrac{K+2}{2}\rho_n^\star\big). \tag{7.57}$$

Now choose $K = \lceil 2\varepsilon_n\big(1-\frac{1}{16}\big)/\rho_n^\star \rceil$. Then $\frac{K+2}{2}\rho_n^\star \ge \varepsilon_n\big(1-\frac{1}{16}\big)$ and, using (2.25),

$$R_n \le n2^{n\varepsilon_n - 2n\rho_n^\star} r_n^{-1}(\varepsilon_n) r_n\big(\varepsilon_n/16\big) \le n2^{-n\varepsilon_n/4 - 2n\rho_n^\star} \tag{7.58}$$

for all $\beta > \beta_c(\varepsilon_n)$. Inserting (7.55) and (7.58) in (7.50),

$$\mathcal{P}_{\pi_n^\circ}\big(\widehat{\mathcal{A}}\big) \le 4\epsilon^{-1}\left(c_0 n^{2-c_\star/\alpha(\varepsilon_n)} + n^{-2(c_\star-1)}2^{-n\varepsilon_n/4}\right) \tag{7.59}$$

for all n large enough. Finally, combining (7.35), (7.45), and (7.59), we obtain that for all $\beta > \beta_c(\varepsilon)$, on $\Omega^\star \cap \Omega_3$, for all but a finite number of indices n,

$$\begin{aligned}\mathcal{P}_{\pi_n^\circ}\big(\widehat{S}_n(1) > \epsilon\big) \;\le\;& e^{-n} + n^{-2(c_\star-1)+c_\circ} + 2n^{-(c_\star-2)/2} \\ &+ 4\epsilon^{-1}\left(c_0 n^{2-c_\star/\alpha(\varepsilon_n)} + n^{-2(c_\star-1)}2^{-n\varepsilon_n/4}\right)\end{aligned} \tag{7.60}$$

for all $\epsilon > 0$, where $c_\circ > 0$ is arbitrarily small, and where $\lim_{n\to\infty}\varepsilon_n = \varepsilon$, $0<\varepsilon<1$. Since by assumption $c_\star > 2$, choosing $\epsilon = n^{-(c_\star-2)/2}$ yields the claim of Lemma 7.7. $\square$

The proof of Theorem 3.3 is now complete.

7.5. **Proof of Theorem 1.3.** Denote respectively by $\widetilde{\mathcal{R}}_n$ and $\mathcal{R}_n$ the ranges of the processes $\widetilde{\mathcal{S}}_n(\cdot) \equiv c_n^{-1}\widetilde{S}_n(\lfloor a_n\cdot\rfloor)$ and $S_n(\cdot) = c_n^{-1}\widetilde{S}_n(K_n(\cdot))$. The set $\widetilde{\mathcal{R}}_n^c = [0,\infty) - \widetilde{\mathcal{R}}_n$ can be decomposed in a canonical way into $\widetilde{\mathcal{R}}_n^c = \cup_{s\in\widetilde{\mathcal{J}}}\big(\widetilde{\mathcal{S}}_n(s_-), \widetilde{\mathcal{S}}_n(s)\big)$ where $\widetilde{\mathcal{J}}$ denotes the set of jump times of $\widetilde{\mathcal{S}}_n$, and $\widetilde{\mathcal{S}}_n(s_-)$ the left limit at s. Clearly, $\widetilde{\mathcal{J}} = \{i/a_n \mid i \in \mathbb{N}\}$. Similarly, $\mathcal{R}_n^c = [0,\infty) - \mathcal{R}_n$ can be decomposed into $\mathcal{R}_n^c = \cup_{s\in\mathcal{J}}\big(\mathcal{S}_n(s_-), \mathcal{S}_n(s)\big)$ where $\mathcal{J}$ denotes the set of jump times of $\mathcal{S}_n$. In view of the definition (1.16) of K_n and (3.10), $\mathcal{J} = \{T^\dagger_{n,i}/a_n \mid i\in\mathbb{N}\} \subseteq \widetilde{\mathcal{J}}$. That is, by construction, $\widetilde{\mathcal{R}}_n^c$ and $\mathcal{R}_n^c$ differ only at the times of visits of X_n to $\mathcal{V}_n^\star$, the increments of $\widetilde{\mathcal{S}}_n$ along the stretches of trajectory that traverse $\mathcal{V}_n^\star$ being lumped into single increments of S_n. In particular, it follows from the definitions (3.4), (3.8), and (3.9) that $\mathcal{J} \cap \widetilde{\mathcal{J}}\{T^\circ_{n,i}/a_n \mid i\in\mathbb{N}\} \equiv \mathcal{J}^\circ$, and

$$\begin{aligned}\widetilde{\mathcal{R}}_n^c - \cup_{s\in\mathcal{J}^\circ}\big(\mathcal{S}_n(s_-), \mathcal{S}_n(s)\big) \;&\subseteq\; \mathcal{R}_n^c - \cup_{s\in\mathcal{J}^\circ}\big(\mathcal{S}_n(s_-), \mathcal{S}_n(s)\big) &(7.61)\\ &=\; \cup_j\big(\mathcal{S}_n(\overline{T}'_{n,j}), \mathcal{S}_n(\overline{T}_{n,j+1}-1)\big). &(7.62)\end{aligned}$$

Next, denoting by m the Lebesgue measure, it follows from (3.24)-(3.25) that

$$m\big(\mathcal{S}_n(\overline{T}'_{n,j}), \mathcal{S}_n(\overline{T}_{n,j+1}-1)\big) \stackrel{d}{=} c_n^{-1}\widehat{\Lambda}^\dagger_n(j) \tag{7.63}$$

where equality holds in distribution. Sums of such terms are of the form $\widehat{S}_n(t')$ (see (3.27)). Now, it follows from Lemma 7.7 that with a probability that goes to 1 as $n\uparrow\infty$, $\widehat{S}_n(t')$

decays to zero as fast as $t'n^{1-c_\star/2}$. This, (7.62), and (7.63) readily implies that for all $0 < T < \infty$,

$$m\big((\widetilde{\mathcal{R}}_n^c - \mathcal{R}_n^c) \cap [0,T)\big) \to 0, \quad n \to \infty, \tag{7.64}$$

Thus, as $n \uparrow \infty$, $\widetilde{\mathcal{R}}_n$ and $\mathcal{R}_n$ coincide on any bounded interval with probability going to 1.

From now on the proof follows classical arguments. Let $A_n^\rho(t,s)$ be the event $A_n^\rho(t,s) \equiv \{\mathcal{C}_n(t,s) \geq 1-\rho\}$. Clearly, for all $\rho \in (0,1)$, $A_n^\rho(t,s) \supseteq \{\widetilde{\mathcal{R}}_n \cap (t,t+s) = \emptyset\}$. By our earlier observations, for all $t,s > 0$, $\{\widetilde{\mathcal{R}}_n \cap (t,t+s) = \emptyset\} = \{\mathcal{R}_n \cap (t,t+s) = \emptyset\}$ with probability going 1. Thus $\lim_{n\to\infty} \mathcal{P}_{\pi_n^\circ}(\widetilde{\mathcal{R}}_n \cap (t,t+s) = \emptyset) = \lim_{n\to\infty} \mathcal{P}_{\pi_n^\circ}(\mathcal{R}_n \cap (t,t+s) = \emptyset)$ and by Theorem 1.1, proceeding as in the proof of Theorem 1.6 in [26], $\lim_{n\to\infty} \mathcal{P}_{\pi_n^\circ}(\mathcal{R}_n \cap (t,t+s) = \emptyset) = \mathcal{P}(\{S_\infty(u), u > 0\} \cap (t,t+s) = \emptyset)$ where, by the arcsine law for stable subordinators (see e.g. Theorem 1.8 of [26]), the last probability is equal to the right hand side of (1.25).

Let us now prove that $\lim_{n\to\infty} \mathcal{P}_{\pi_n^\circ}\big(A_n^\rho(t,s) \cap \{\widetilde{\mathcal{R}}_n \cap (t,t+s) \neq \emptyset\}\big) = 0$. Invoking as before Lemma 7.7, we can substitute $\mathcal{R}$ for $\widetilde{\mathcal{R}}_n$ in the above probability. Consider the set $\mathcal{T}_n(\epsilon) \equiv \{x \in I_n^\star \mid w_n(x) \geq \epsilon c_n\}$, $\epsilon > 0$. By Theorem 1.1, if $\mathcal{R}_n \cap (t,t+s) \neq \emptyset$ then with a probability that tends to one as $n \uparrow \infty$ and $\epsilon \downarrow 0$ there exists $u_- \leq u_+$ such that on the one hand $c_n^{-1}\widetilde{S}_n(K_n(u_-)-1) < t < c_n^{-1}\widetilde{S}_n(K_n(u_-))$ while $c_n^{-1}\widetilde{S}_n(K_n(u_+)) < t < c_n^{-1}\widetilde{S}_n(K_n(u_+)+1)$ on the other, and these two increments correspond to visits to vertices z_- and z_+ in $\mathcal{T}_n(\epsilon)$ (that is to say, with probability one, the points t and $t+s$ lie in constancy intervals of the process, and such intervals are produced, asymptotically, by visits to $\mathcal{T}_n(\epsilon)$). Let us now argue that, firstly, starting from a given vertex $z_- \in \mathcal{T}_n(\epsilon)$, the chain J_n° quickly moves at a distance greater than $n\rho/2$ from it, and secondly, that it does not visit any vertex in $\{z \in \mathcal{T}_n(\epsilon) \mid \mathrm{dist}(z_-,z) \leq n\rho/2\}$ in the ensuing $\lfloor Ca_n \rfloor$ steps, for any $0 < C < \infty$, $0 < \rho < 1$, and small $\epsilon > 0$. For this we use three results of Section 6. By Proposition 6.1, the chain J_n° started in z_- reaches stationarity in $\ell_n^\circ \sim n^{2(c_\star+1)}/(\log n)^2$ steps, and by Proposition 6.3, $\pi_n^\circ(\{z \in \mathcal{V}_n^\circ \mid \mathrm{dist}(z_-,z) > n\rho/2\}) \geq 1 - \exp\{-n\mathcal{I}(\rho)\}$, where $\mathcal{I}(\rho) > 0$ if $0 < \rho < 1$. This proves the first claim. The second claim is an immediate consequence of Proposition 6.4. The proof of Theorem 1.3 is done.

8. Convergence of the front end clock process: Proof of Theorem 3.2

The proofs of Theorem 3.2 and Theorem 3.4 rely on a method developped by Durrett and Resnick [22] that provides sufficient conditions for partial sum processes to converge to Lévy processes. We use their results in a specialized form suitable for our applications which is taken from [26], where this method was first applied to the study of clock processes in random environment; see also [14] where it was implemented in more generality.

8.1. A convergence theorem for FECP.

Consider the rescaled front end clock process (3.26),

$$S_n^\circ(t) = c_n^{-1}\widetilde{S}_n^\circ(\lfloor a_n t \rfloor), \quad t \geq 0. \tag{8.1}$$

Theorem 8.1 below is the corner stone of the proof of Theorem 3.2. It deduces convergence of S_n° to a subordinator from a set of four conditions which we now formulate. Note that these conditions refer to given sequences of numbers a_n and c_n, as well as a given realization of the random environment. For $t > 0$ and $u > 0$ define

$$h_n^u(y) = \sum_{x\in\mathcal{V}_n^\circ} p_n^\circ(y,x)\exp\{-uc_n\lambda_n(x)\}\ ,\ y \in \mathcal{V}_n^\circ, \tag{8.2}$$

and, recalling the notation $k_n^\circ(t) = \lfloor a_n t \rfloor$,

$$\nu_n^{J_n^\circ,t}(u,\infty) = \sum_{j=0}^{k_n^\circ(t)-1} h_n^u(J_n^\circ(j)), \tag{8.3}$$

$$\sigma_n^{J_n^\circ,t}(u,\infty) = \sum_{j=0}^{k_n^\circ(t)-1} \left[h_n^u(J_n^\circ(j))\right]^2. \tag{8.4}$$

Condition (C0). For all $v > 0$,

$$\textstyle\sum_{x\in\mathcal{V}_n^\circ} \pi_n^\circ(x) e^{-vc_n\lambda_n(x)} = o(1). \tag{8.5}$$

Condition (C1). There exists a σ-finite measure ν° on $(0,\infty)$ satisfying $\int_0^\infty (1\wedge u)\nu^\circ(du) < \infty$ such that, for all $t > 0$ and all $u > 0$,

$$P_{\pi_n^\circ}^\circ\left(\left|\nu_n^{J_n^\circ,t}(u,\infty) - t\nu^\circ(u,\infty)\right| < \epsilon\right) = 1 - o(1), \quad \forall \epsilon > 0. \tag{8.6}$$

Condition (C2). For all $u > 0$ and all $t > 0$,

$$P_{\pi_n^\circ}^\circ\left(\sigma_n^{J_n^\circ,t}(u,\infty) < \epsilon\right) = 1 - o(1), \quad \forall \epsilon > 0. \tag{8.7}$$

Condition (C3). For all $t > 0$,

$$\lim_{\epsilon\downarrow 0}\limsup_{n\uparrow\infty} k_n^\circ(t)\mathcal{E}_{\pi_n^\circ}^\circ \mathbb{1}_{\{\lambda_n^{-1}(J_n^\circ(0))e_0^\circ \le c_n\epsilon\}} c_n^{-1}\lambda_n^{-1}(J_n^\circ(0))e_0^\circ = 0. \tag{8.8}$$

Theorem 8.1. *Let the initial distribution of J_n° be its invariant measure π_n°. For all sequences a_n and c_n for which Conditions (C0), (C1), (C2), and (C3) are verified $\mathbb{P}$-almost surely,*

$$S_n^\circ \Rightarrow_{J_1} S_\infty^\circ \tag{8.9}$$

$\mathbb{P}$-almost surely, where S_∞° is the Lévy subordinator with zero drift and Lévy measure ν°.

Proof. This is a restatement of Theorem 1.2 of [14] specialized to the case where θ_n, the "bloc length", is equal to one. (Theorem 1.2 of [14] is itself a generalization of Theorem 1.1 of [26] with a more workable Condition (C3).) □

To verify the conditions of Theorem 8.1 we follow a by now well established two-step strategy that was first proposed in [25], and was used later in [14]. The first step consists in using the mixing property and mean local time estimates of Proposition 6.1 and Proposition 6.2, respectively, to prove an almost sure ergodic theorem for the quantities (8.3) and (8.4). This is done in Subsection 8.2 (see Theorem 8.2). It then enables us to reduce Conditions (C1) and (C2) of Theorem 8.1 to laws of large numbers in the random environment. This second step is carried out in Subsection 8.3 (see Proposition 8.5). The proof is completed in Subsection 8.4.

8.2. **An ergodic theorem for FECP.** Let $\pi_n^{J_n^\circ,t}(x)$ denote the average number of visits of J_n° to x during the first $k_n^\circ(t)$ steps,

$$\pi_n^{J_n^\circ,t}(x) = (k_n^\circ(t))^{-1} \sum_{j=0}^{k_n^\circ(t)-1} \mathbb{1}_{\{J_n^\circ(j)=x\}}, \quad x \in \mathcal{V}_n^\circ. \tag{8.10}$$

Then (8.3) and (8.4) can be rewritten as

$$\nu_n^{J_n^\circ,t}(u,\infty) = k_n^\circ(t)\sum_{y\in\mathcal{V}_n^\circ}\pi_n^{J_n^\circ,t}(y)h_n^u(y), \tag{8.11}$$

$$\sigma_n^{J_n^\circ,t}(u,\infty) = k_n^\circ(t)\sum_{y\in\mathcal{V}_n^\circ}\pi_n^{J_n^\circ,t}(y)\left[h_n^u(y)\right]^2. \tag{8.12}$$

One readily sees, using reversibility, that

$$E_{\pi_n^\circ}\left[\nu_n^{J_n^\circ,t}(u,\infty)\right] = k_n^\circ(t)\sum_{x\in\mathcal{V}_n^\circ}\pi_n^\circ(x)h_n^u(x) = (k_n^\circ(t)/a_n)\nu_n^\circ(u,\infty), \tag{8.13}$$

$$E_{\pi_n^\circ}\left[\sigma_n^{J_n^\circ,t}(u,\infty)\right] = k_n^\circ(t)\sum_{x\in\mathcal{V}_n^\circ}\pi_n^\circ(x)\left[h_n^u(x)\right]^2 = (k_n^\circ(t)/a_n)\sigma_n^\circ(u,\infty), \tag{8.14}$$

where

$$\nu_n^\circ(u,\infty) = \frac{a_n}{|\mathcal{V}_n^\circ|}\sum_{x\in\mathcal{V}_n^\circ}e^{-uc_n\lambda_n(x)}, \tag{8.15}$$

$$\sigma_n^\circ(u,\infty) = \frac{a_n}{|\mathcal{V}_n^\circ|}\sum_{x\in\mathcal{V}_n^\circ}\sum_{x'\in\mathcal{V}_n^\circ}p_n^{\circ,2}(x,x')e^{-uc_n(\lambda_n(x)+\lambda_n(x'))}. \tag{8.16}$$

Here $p_n^{\circ,2}(\cdot,\cdot)$ denotes the 2-steps transition probabilities of J_n°. Note that since $H(x)=0$ for all $x\in\mathcal{V}_n^\circ\setminus I_n^\star$, where $I_n^\star$ is the set of isolated vertices in the partition (2.7), we have

$$\lambda_n(x) = \begin{cases} e^{\beta H_n(x)}, & \text{if } x\in I_n^\star,\\ 1, & \text{if } x\in\mathcal{V}_n^\circ\setminus I_n^\star.\end{cases} \tag{8.17}$$

Theorem 8.2. *Assume that $c_\star>3$. Let $\rho_n^\circ>0$ be a decreasing sequence satisfying $\rho_n^\circ\downarrow 0$ as $n\uparrow\infty$. There exists a sequence of subsets $\Omega_n^{EG}\subset\Omega$ with $\mathbb{P}\left[(\Omega_n^{EG})^c\right]<\ell_n^\circ/(\rho_n^\circ a_n)+n^{-2}$, for ℓ_n° as in (6.1), and such that on Ω_n^{EG} the following holds for all large enough n: for all $t>0$, all $u>0$, and all $\epsilon>0$,*

$$P_{\pi_n^\circ}\left(\left|\nu_n^{J_n^\circ,t}(u,\infty)-(k_n^\circ(t)/a_n)\nu_n^\circ(u,\infty)\right|\geq\epsilon\right)\leq\epsilon^{-2}[C_1t\Theta_{n,1}(u)+t^2\Theta_{n,2}(u)] \tag{8.18}$$

for some constant $0<C_1<\infty$, where

$$\Theta_{n,1}(u)\equiv\ell_n^\circ e^{-uc_n}[1+\nu_n^\circ(u,\infty)]+\sigma_n^\circ(u,\infty)+\frac{\nu_n^\circ(2u,\infty)}{n\log n}+\rho_n^\circ\left[\mathbb{E}\nu_n^\circ(u,\infty)\right]^2, \tag{8.19}$$

$$\Theta_{n,2}(u)\equiv 2^{-n}\left[\nu_n^\circ(u,\infty)\right]^2. \tag{8.20}$$

Moreover, for all $t>0$, all $u>0$, and all $\epsilon'>0$,

$$P_{\pi_n^\circ}\left(\sigma_n^{J_n^\circ,t}(u,\infty)\geq\epsilon'\right)\leq\frac{t}{\epsilon'}(1+o(1))\sigma_n^\circ(u,\infty). \tag{8.21}$$

Proof of Theorem 8.2 . The upper bound (8.21) simply results from a first order Tchebychev inequality and (8.14). The proof of (8.18) is more involved. It relies on a second order Tchebychev inequality, that is, using (8.13), we bound the left hand side of (8.18) from above by

$$\epsilon^{-2}(k_n^\circ(t))^2\sum_{x\in\mathcal{V}_n^\circ}\sum_{y\in\mathcal{V}_n^\circ}h_n^u(x)h_n^u(y)E_{\pi_n^\circ}\left(\pi_n^{J_n^\circ,t}(x)-\pi_n^\circ(x)\right)\left(\pi_n^{J_n^\circ,t}(y)-\pi_n^\circ(y)\right). \tag{8.22}$$

In view of (8.10), setting $\Delta_{ij}(x,y)=P_{\pi_n^\circ}\left(J_n^\circ(i)=x,J_n^\circ(j)=y\right)-\pi_n^\circ(x)\pi_n^\circ(y)$, the expectation in (8.22) may be rewritten as

$$E_{\pi_n^\circ}\left(\pi_n^{J_n^\circ,t}(x)-\pi_n^\circ(x)\right)\left(\pi_n^{J_n^\circ,t}(y)-\pi_n^\circ(y)\right)=\sum_{i=0}^{k_n^\circ(t)-1}\sum_{j=0}^{k_n^\circ(t)-1}\Delta_{ij}(x,y). \tag{8.23}$$

For ℓ_n° defined in (6.1) we now break the sum in the r.h.s. of (8.23) into three terms:

$$\begin{aligned} I_1^{(1)} &= 2\sum_{0\le i\le k_n^\circ(t)-1}\sum_{i+\ell_n^\circ\le j\le k_n^\circ(t)-1}\Delta_{ij}(x,y)\,,\\ I_2^{(1)} &= \sum_{0\le i\le k_n^\circ(t)-1}\mathbb{1}_{\{i=j\}}\Delta_{ij}(x,y)\,,\\ I_3^{(1)} &= 2\sum_{0\le i\le k_n^\circ(t)-1}\sum_{i<j<i+\ell_n^\circ}\Delta_{ij}(x,y)\,. \end{aligned} \tag{8.24}$$

Consider first $I_1^{(1)}$. By Proposition 6.1,

$$I_1^{(1)} \le \delta_n(k_n^\circ(t))^2\pi_n^\circ(x)\pi_n^\circ(y) \le 2^{-n}(k_n^\circ(t))^2\pi_n^\circ(x)\pi_n^\circ(y). \tag{8.25}$$

Turning to the term $I_2^{(1)}$, we have,

$$I_2^{(1)} = \sum_{1\le i\le k_n^\circ(t)}\Delta_{ii}(x,x)\mathbb{1}_{\{x=y\}} = k_n^\circ(t)\pi_n^\circ(x)(1-\pi_n^\circ(x))\mathbb{1}_{\{x=y\}}, \tag{8.26}$$

where we used that $P^\circ_{\pi_n^\circ}(J_n^\circ(i)=x)=\pi_n^\circ(x)$. Finally,

$$\begin{aligned} I_3^{(1)} &\le 2\sum_{i=0}^{k_n^\circ(t)-1}\sum_{l=1}^{\ell_n^\circ-1}P^\circ_{\pi_n^\circ}\left(J_n^\circ(i)=x, J_n^\circ(i+l)=y\right)\\ &= 2k_n^\circ(t)\pi_n^\circ(x)\sum_{l=1}^{\ell_n^\circ-1}p_n^{\circ,l}(x,y) \end{aligned} \tag{8.27}$$

where $p_n^{\circ,l}(\cdot,\cdot)$ denote the l-steps transition probabilities of J_n°. Combining our bounds on $(\overline{I})$, $I_2^{(1)}$, and $I_3^{(1)}$ with (8.22) we get that, for all $\epsilon>0$,

$$P^\circ_{\pi_n^\circ}\left(\left|\nu_n^{J_n^\circ,t}(u,\infty)-E^\circ_{\pi_n^\circ}\left[\nu_n^{J_n^\circ,t}(u,\infty)\right]\right|\ge\epsilon\right)\le\epsilon^{-2}[I_1^{(2)}+I_2^{(2)}+I_3^{(2)}]\,, \tag{8.28}$$

where

$$\begin{aligned} I_1^{(2)} &= 2^{-n}(k_n^\circ(t))^2\sum_{x\in\mathcal{V}_n^\circ}\sum_{y\in\mathcal{V}_n^\circ}h_n^u(x)h_n^u(y)\pi_n^\circ(x)\pi_n^\circ(y)\,,\\ I_2^{(2)} &= k_n^\circ(t)\sum_{x\in\mathcal{V}_n^\circ}\sum_{y\in\mathcal{V}_n^\circ}h_n^u(x)h_n^u(y)\pi_n^\circ(x)(1-\pi_n^\circ(x))\mathbb{1}_{\{x=y\}}\,,\\ I_3^{(2)} &= 2k_n^\circ(t)\sum_{x\in\mathcal{V}_n^\circ}\sum_{y\in\mathcal{V}_n^\circ}h_n^u(x)h_n^u(y)\pi_n^\circ(x)\sum_{l=1}^{\ell_n^\circ-1}p_n^{\circ,l}(x,y). \end{aligned} \tag{8.29}$$

In view of (8.13)-(8.14),

$$I_1^{(2)} \le 2^{-n}\left(k_n^\circ(t)/a_n\right)^2\left[\nu_n^\circ(u,\infty)\right]^2\,, \tag{8.30}$$

$$I_2^{(2)} \le (k_n^\circ(t)/a_n)\sigma_n^\circ(u,\infty). \tag{8.31}$$

To deal with the third term in (8.29) note first that by (8.2),

$$\sum_{y\in\mathcal{V}_n^\circ}p_n^{\circ,l}(x,y)h_n^u(y) = \sum_{z\in\mathcal{V}_n^\circ}p_n^{\circ,l+1}(x,z)e^{-uc_n\lambda_n(z)}, \tag{8.32}$$

so that

$$\begin{aligned} \sum_{x\in\mathcal{V}_n^\circ}\pi_n^\circ(x)h_n^u(x)p_n^{\circ,l+1}(x,z) &= \sum_{y\in\mathcal{V}_n^\circ}e^{-uc_n\lambda_n(y)}\sum_{x\in\mathcal{V}_n^\circ}\pi_n^\circ(x)p_n^\circ(x,y)p_n^{\circ,l+1}(x,z)\\ &= \sum_{y\in\mathcal{V}_n^\circ}e^{-uc_n\lambda_n(y)}\pi_n^\circ(y)p_n^{\circ,l+2}(y,z)\,, \end{aligned} \tag{8.33}$$

where the last equality follows by reversibility. Hence,

$$
\begin{aligned}
I_3^{(2)} &= 2k_n^\circ(t)\sum_{l=1}^{\ell_n^\circ-1}\sum_{z\in\mathcal{V}_n^\circ}\Big[\sum_{x\in\mathcal{V}_n^\circ}\pi_n^\circ(x)h_n^u(x)p_n^{\circ,l+1}(x,z)\Big]e^{-uc_n\lambda_n(z)}\,,\\
&= 2\sum_{l=1}^{\ell_n^\circ-1}k_n^\circ(t)\sum_{z\in\mathcal{V}_n^\circ}\sum_{y\in\mathcal{V}_n^\circ}\pi_n^\circ(y)e^{-uc_n(\lambda_n(y)+\lambda_n(z))}p_n^{\circ,l+2}(y,z)\\
&\equiv 2(k_n^\circ(t)/a_n)\sum_{z\in\mathcal{V}_n^\circ}\sum_{y\in\mathcal{V}_n^\circ}f_n(y,z)
\end{aligned}
\tag{8.34}
$$

where the last line defines $f_n(y,z)$. In view of (8.17), we have

$$
\sum_{z\in\mathcal{V}_n^\circ\setminus I_n^\star}\sum_{y\in\mathcal{V}_n^\circ\setminus I_n^\star}f_n(y,z)\le \ell_n^\circ e^{-2uc_n},
\tag{8.35}
$$

$$
\sum_{z\in\mathcal{V}_n^\circ\setminus I_n^\star}\sum_{y\in I_n^\star}f_n(y,z)=\sum_{z\in I_n^\star}\sum_{y\in\mathcal{V}_n^\circ\setminus I_n^\star}f_n(y,z)\le \ell_n^\circ e^{-uc_n}\nu_n^\circ(u,\infty),
\tag{8.36}
$$

where the equality above is reversibility. It thus remains to bound the term

$$
I^{(3)}\equiv 2\frac{k_n^\circ(t)}{a_n}\sum_{z\in I_n^\star}\sum_{y\in I_n^\star}f_n(y,z)=2\frac{k_n^\circ(t)}{a_n}\sum_{l=1}^{\ell_n^\circ-1}[I_{1,l}^{(3)}+I_{2,l}^{(3)}+I_{3,l}^{(3)}],
\tag{8.37}
$$

where, distinguishing the cases $z=y$ and $z\neq y$, and for $\mathcal{W}_n$ defined in (6.3),

$$
I_{1,l}^{(3)}\equiv\sum_{z\in I_n^\star\cap\mathcal{W}_n}a_n\pi_n^\circ(z)e^{-2uc_n\lambda_n(z)}p_n^{\circ,l+2}(z,z),
\tag{8.38}
$$

$$
I_{2,l}^{(3)}\equiv\sum_{z\in I_n^\star\setminus\mathcal{W}_n}a_n\pi_n^\circ(z)e^{-2uc_n\lambda_n(z)}p_n^{\circ,l+2}(z,z),
\tag{8.39}
$$

$$
I_{3,l}^{(3)}\equiv\sum_{z\in I_n^\star}\sum_{y\in I_n^\star:y\neq z}a_n\pi_n^\circ(y)e^{-uc_n(\lambda_n(y)+\lambda_n(z))}p_n^{\circ,l+2}(y,z).
\tag{8.40}
$$

By Proposition 6.2 we have that, on $\Omega^{\mathsf{SRW}}\cap\Omega^\star$, for all but a finite number of indices n,

$$
\sum_{l=1}^{\ell_n^\circ-1}I_{1,l}^{(3)}\le\frac{C_\circ}{\log n}\nu_n^\circ(2u,\infty).
\tag{8.41}
$$

for some constant $0<C_\circ<\infty$. The next two lemmata are designed to deal with (8.39) and (8.40), respectively.

Lemma 8.3. *There exists a sequence of subsets $\Omega_{2,n}^{(3)}\subset\Omega$ with $\mathbb{P}\big(\Omega_{2,n}^{(3)}\big)\ge 1-n^{-2}$ such that on $\Omega_{2,n}^{(3)}$, if $\kappa_\star=[(c_\star+2)/(c_\star-1)]^2$,*

$$
\sum_{l=1}^{\ell_n^\circ-1}I_{2,l}^{(3)}<\frac{1}{\log n}\mathbb{E}\nu_n^\circ(2u,\infty).
\tag{8.42}
$$

Proof. By (6.3),

$$
\begin{aligned}
&\mathbb{E}\textstyle\sum_{l=1}^{\ell_n^\circ-1}I_{2,l}^{(3)}\\
&\le \ell_n^\circ\mathbb{E}\textstyle\sum_{z\in\mathcal{V}_n}a_n\pi_n^\circ(z)e^{-2uc_n\lambda_n(z)}\left(\mathbb{1}_{\left\{\sum_{l=1}^{L^\star}|\partial z\cap\partial C_{n,l}^\star|>\kappa_\star\right\}}+\mathbb{1}_{\left\{\sum_{l=1}^{L^\star}|\partial_2 z\cap\partial C_{n,l}^\star|>\frac{\log n}{n}\right\}}\right)\\
&\le \ell_n^\circ\mathbb{E}\nu_n^\circ(2u,\infty)\left[\mathbb{P}(\textstyle\sum_{l=1}^{L^\star}|\partial z\cap\partial C_{n,l}^\star|\ge\kappa_\star)+\mathbb{P}(\sum_{l=1}^{L^\star}|\partial_2 z\cap\partial C_{n,l}^\star|>\frac{\log n}{n})\right].
\end{aligned}
\tag{8.43}
$$

Thus, by a first order Tchebychev inequality, (8.43), and (2.16)-(2.17), for all $\eta > 0$,

$$\mathbb{P}\big(\textstyle\sum_{l=1}^{\ell_n^\circ-1} I_{2,l}^{(3)} \geq \eta\big) \leq \eta^{-1}\ell_n^\circ(2n^{-\sqrt{\kappa_\star}(2c_\star-3)} + e^{-(2c_\star-3)\sqrt{n\log n}})\mathbb{E}\nu_n^\circ(2u,\infty). \tag{8.44}$$

In view of (6.1), choosing $\kappa_\star = [(c_\star+2)/(c_\star-1)]^2$ and $\eta = \mathbb{E}\nu_n^\circ(2u,\infty)/\log n$ yields the claim of the lemma. □

Lemma 8.4. *Let $\rho_n^\circ > 0$ be a decreasing sequence satisfying $\rho_n^\circ \downarrow 0$ as $n \uparrow \infty$. There exists a sequence of subsets $\Omega_{3,n}^{(3)} \subset \Omega$ with $\mathbb{P}\big(\Omega_{3,n}^{(3)}\big) \geq 1 - \ell_n^\circ/(\rho_n^\circ a_n)$ such that on $\Omega_{3,n}^{(3)}$,*

$$\sum_{l=1}^{\ell_n^\circ-1} I_{3,l}^{(3)} < \rho_n^\circ\left[\mathbb{E}\nu_n^\circ(u,\infty)\right]^2. \tag{8.45}$$

Proof. By definition of $I_n^\star$, $\mathrm{dist}(y,z) \geq 2$ for all $y \in I_n^\star$ and $z \in I_n^\star$ such that $y \neq z$. Thus

$$\begin{aligned}\mathbb{E}\left(e^{-uc_n(\lambda_n(y)+\lambda_n(z))}\mathbb{1}_{\{y\in I_n^\star, z\in I_n^\star\}}\right)\mathbb{1}_{\{x\neq z\}} &\leq \mathbb{E}\left(e^{-uc_n(\lambda_n(y)+\lambda_n(z))}\right)\mathbb{1}_{\{\mathrm{dist}(y,z)\geq 2\}} && (8.46)\\ &\leq \left(\mathbb{E}e^{-uc_n\lambda_n(y)}\right)\left(\mathbb{E}e^{-uc_n\lambda_n(z)}\right) && (8.47)\\ &= \left[a_n^{-1}\mathbb{E}\nu_n^\circ(u,\infty)\right]^2 && (8.48)\end{aligned}$$

where we used independence in the second line. Therefore, by a first order Tchebychev inequality, for all $\eta > 0$,

$$\begin{aligned}\mathbb{P}\big(\textstyle\sum_{l=1}^{\ell_n^\circ-1} I_{3,l}^{(3)} \geq \eta\big) &\leq \frac{1}{\eta a_n}\left[\mathbb{E}\nu_n(u,\infty)\right]^2\sum_{l=1}^{\ell_n^\circ-1}\sum_{y\in\mathcal{V}_n^\circ}\pi_n^\circ(y)\sum_{z\in\mathcal{V}_n^\circ}p_n^{\circ,l+2}(y,z) && (8.49)\\ &\leq \frac{\ell_n^\circ}{\eta a_n}\left[\mathbb{E}\nu_n^\circ(u,\infty)\right]^2. && (8.50)\end{aligned}$$

The lemma now easily follows. □

Gathering our bounds we conclude that under the assumptions of Proposition 6.2, Lemma 8.4, and Lemma 8.3, on $\Omega^{\mathsf{SRW}} \cap \Omega_{2,n}^{(3)} \cap \Omega_{3,n}^{(3)}$, for all but a finite number of indices n,

$$I_3^{(2)} \leq 2\frac{k_n^\circ(t)}{a_n}\left[\ell_n^\circ e^{-uc_n}[1+\nu_n^\circ(u,\infty)] + C_\circ\frac{\nu_n^\circ(2u,\infty)}{\log n} + \rho_n^\circ\left[\mathbb{E}\nu_n^\circ(u,\infty)\right]^2\right] \tag{8.51}$$

for some constant $0 < C_\circ < \infty$. Inserting the bounds (8.30), (8.31), and (8.51) in (8.28) now yields (8.18)-(8.20). The proof of Theorem 8.2 is done. □

8.3. **Almost sure convergence of ν_n° and σ_n°.** Theorem 8.2 enables us to replace the chain dependant quantities $\nu_n^{J_n^\circ,t}$ and $\sigma_n^{J_n^\circ,t}$ by quantities, ν_n° and σ_n°, that now only depend on the randomness of the environment. Our next step consists in proving laws of large numbers for ν_n° and σ_n°.

Proposition 8.5. *Under the assumptions and with the notation of Theorem 1.1 there exists a subset $\Omega^{\mathsf{LLN}} \subset \Omega$ with $\mathbb{P}(\Omega^{\mathsf{LLN}}) = 1$ such that, on Ω^{LLN}, the following holds: for all $u > 0$,*

$$\begin{aligned}\lim_{n\to\infty}\nu_n^\circ(u,\infty) &= \nu(u,\infty), && (8.52)\\ \lim_{n\to\infty} n\sigma_n^\circ(u,\infty) &= \nu(2u,\infty). && (8.53)\end{aligned}$$

We prove the proposition by comparing ν_n° and σ_n° to their counterpart, ν_n^{REM} and σ_n^{REM}, in the random hopping dynamics of the non truncated REM. To define ν_n^{REM} and σ_n^{REM} recall the definition of $w_n(x)$ from (1.15) and set

$$\gamma_n(x) = w_n(x)/c_n. \tag{8.54}$$

Then, for all $u > 0$,

$$\nu_n^{\mathsf{REM}}(u,\infty) = \frac{a_n}{|\mathcal{V}_n|}\sum_{x\in\mathcal{V}_n} e^{-u/\gamma_n(x)}, \tag{8.55}$$

$$\sigma_n^{\mathsf{REM}}(u,\infty) = \frac{a_n}{|\mathcal{V}_n|}\sum_{x\in\mathcal{V}_n}\sum_{x'\in\mathcal{V}_n} p_n^{\mathsf{SRW},(2)}(x,x')e^{-u(1/\gamma_n(x)+1/\gamma_n(x'))}, \tag{8.56}$$

where $p_n^{\mathsf{SRW},(2)}(\cdot,\cdot)$ denotes the 2-steps transition probabilities of J_n^{SRW} (see (6.48)). For later use (namely, for the treatment of Condition (C3)) we also define, for all $\epsilon > 0$,

$$\eta_n^{\mathsf{REM}}(\epsilon) = \frac{a_n}{|\mathcal{V}_n|}\sum_{x\in\mathcal{V}_n}\gamma_n(x)\left(1-e^{-\epsilon/\gamma_n(x)}\right). \tag{8.57}$$

The functions ν_n^{REM}, σ_n^{REM}, and η_n^{REM} are well understood. We know in particular that:

Proposition 8.6. *Given $0 < \varepsilon < 1$ let a_n and c_n be as in Theorem 1.1. Let ν be as in (1.21) and assume that $\beta > \beta_c(\varepsilon)$. Then, there exists a subset $\Omega^{REM} \subset \Omega$ with $\mathbb{P}(\Omega^{REM}) = 1$ such that, on Ω^{REM}, the following holds:*

$$\lim_{n\to\infty}\nu_n^{REM}(u,\infty) = \nu(u,\infty), \quad \forall u > 0, \tag{8.58}$$

$$\lim_{n\to\infty} n\sigma_n^{REM}(u,\infty) = \nu(2u,\infty), \quad \forall u > 0, \tag{8.59}$$

and

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\eta_n^{REM}(\epsilon) = 0. \tag{8.60}$$

Throughout this section we set $\varepsilon_n \equiv \frac{\log a_n}{n\log 2}$; thus by 1.19, $\lim_{n\to\infty}\varepsilon_n = \varepsilon$, $0<\varepsilon<1$.

Proof. Eq. (8.58) and (8.59) are proved in Proposition 5.1, (i), in Section 5.1 of [25]. The proof of (8.60) is elementary: by simple Gaussian calculations, $\mathbb{E}\eta_n^{\mathsf{REM}}(\epsilon) \le c\epsilon^{1-\alpha(\varepsilon_n)} \downarrow 0$ as $n\uparrow\infty$ and $\epsilon\downarrow 0$, where $0<c<\infty$ is a constant, and $\mathbb{P}\left(|\eta_n^{\mathsf{REM}}(\epsilon)-\mathbb{E}\eta_n^{\mathsf{REM}}(\epsilon)| > n^{-1}\right) \le n^3 a_n/|\mathcal{V}_n|$, which is summable under our assumptions on a_n. Since $\eta_n^{\mathsf{REM}}(\epsilon)$ is a monotonic function of $\epsilon > 0$, arguing e.g. as in (9.119) yields the claim (8.60). $\square$

Our next lemma establishes that ν_n° and σ_n° are very close to ν_n^{REM} and σ_n^{REM}.

Lemma 8.7. *On Ω_3, for all but a finite number of indices n, for all $u > 0$,*

$$|\nu_n^\circ(u,\infty)-\nu_n^{REM}(u,\infty)| \le 2n^{-2c_\star+1}\nu_n^{REM}(u,\infty)+2a_ne^{-un^2}+2n^{-c_\star+1+2\alpha(\varepsilon_n)}, \tag{8.61}$$

$$|\sigma_n^\circ(u,\infty)-\sigma_n^{REM}(u,\infty)| \le 2n^{-2c_\star+1}\sigma_n^{REM}(u,\infty)+4a_ne^{-un^2}+2n^{-c_\star+1+2\alpha(\varepsilon_n)}. \tag{8.62}$$

Proof of Lemma 8.7 . The proof hinges on the observation that $c_n\lambda_n(x) = 1/\gamma_n(x)$ for all x in the subset $I_n^\star$ of the decomposition (2.7). This enables us to rewrite $\nu_n^\circ(u,\infty)$ as

$$\nu_n^\circ(u,\infty) = (|\mathcal{V}_n|/|\mathcal{V}_n^\circ|)\nu_n^{\mathsf{REM}}(u,\infty)+I_1-I_2-I_3 \tag{8.63}$$

where

$$I_3 \equiv (a_n/|\mathcal{V}_n^\circ|)\textstyle\sum_{x\in\cup_{l=1}^{L^\star}C_{n,l}^\star} e^{-u/\gamma_n(x)}, \tag{8.64}$$

$$I_1 \equiv (a_n/|\mathcal{V}_n^\circ|)\textstyle\sum_{x\in\mathcal{V}_n^\circ\setminus I_n^\star} e^{-uc_n\lambda_n(x)} \le a_ne^{-uc_n}, \tag{8.65}$$

$$I_2 \equiv (a_n/|\mathcal{V}_n^\circ|)\textstyle\sum_{x\in\mathcal{V}_n^\circ\setminus I_n^\star} e^{-u/\gamma_n(x)} \le a_ne^{-uc_n/r_n(\rho_n^\star)}. \tag{8.66}$$

The bounds on I_1 and I_2 follow from the fact that on $\mathcal{V}_n^\circ\setminus I_n^\star \equiv N_n^\star$, $\lambda_n(x) = 1$ and $w_n(x) < r_n(\rho_n^\star)$. In order to bound I_3 recall 2.1 and set $\mathcal{W}_n(\rho) \equiv (\cup_{l=1}^{L^\star}C_{n,l}^\star)\cap V(\rho)$ and $\mathcal{W}_n^c(\rho) \equiv (\cup_{l=1}^{L^\star}C_{n,l}^\star)\cap V^c(\rho)$ for some $\rho > 0$. Then, on $\mathcal{W}_n^c(\rho)$, by (2.25) of Lemma 2.3,

$$\tfrac{w_n(x)}{c_n} \le \tfrac{r_n(\rho)}{r_n(\varepsilon_n)} = \exp\{n\beta\beta_c(1)(\sqrt{\varepsilon}_n-\sqrt{\rho})-\tfrac{\beta\log n}{2\beta_c(1)}(\tfrac{1}{\sqrt{\varepsilon}_n}-\tfrac{1}{\sqrt{\rho}})+o(1)\}, \tag{8.67}$$

so that choosing $\sqrt{\rho} = \sqrt{\varepsilon}_n - \frac{2\log n}{n\beta\beta_c(1)}$, we get

$$\frac{r_n(\rho)}{r_n(\varepsilon_n)} = n^2 \exp\left\{\frac{\log n}{n\beta\varepsilon_n 2\log 2}(1+o(1))\right\} = n^2(1+o(1)). \tag{8.68}$$

One also sees that for this choice of ρ, $4\rho_n^\star < \rho < 1 - 4\rho_n^\star$ for all $0 < \varepsilon < 1$ and large enough n. Therefore Lemma 7.1 applies, yielding

$$|\mathcal{W}_n(\rho)| / |\mathcal{V}_n^\circ| \leq n^{-c_\star+1} 2^{-n\rho}(1+o(1)), \tag{8.69}$$

on Ω_3, for all n large enough. Assume from now on that $\omega \in \Omega_3$. By (8.68) and (8.69),

$$I_3 \leq a_n e^{-un^2} + 2n^{-c_\star+1} 2^{n(\varepsilon_n - \rho)} \leq a_n e^{-un^2} + 2n^{-c_\star+1} n^{2\beta_c(\varepsilon_n)/\beta}, \tag{8.70}$$

where we used that $\varepsilon_n - \rho = (\sqrt{\varepsilon_n} - \sqrt{\rho})(\sqrt{\varepsilon_n} + \sqrt{\rho}) \leq 2\sqrt{\varepsilon_n}\frac{2\log n}{n\beta\beta_c(1)}$. Eq. (8.61) now easily follows observing that, by (2.25) and (2.26) of Lemma 2.3, $c_n \gg c_n / r_n(\rho_n^\star) \gg n^2$, and using that $|\mathcal{V}_n| / |\mathcal{V}_n^\circ| = 1 + n^{-2c_\star+1}(1 + \mathcal{O}(n^{-(c_\star-1)}))$, as follows from (6.7).

The proof of (8.62) follows the same pattern, using the additionnal observation that $p_n^{\circ,2}(x,x') = p_n^{\mathsf{SRW},2}(x,x')$ for all x, x' in $I_n^\star \times I_n^\star$. This follows from Proposition 6.5 and the fact that, by construction, $I_n^\star \cap \partial C_{n,l}^\star = \emptyset$ for all $1 \leq l \leq L^\star$. We skip the details. □

Proof of Proposition 8.5 . The proposition is now an immediate consequence of Lemma 8.7 and Proposition 8.6. □

8.4. **Conclusion of the proof of Theorem 3.2.** We are now ready to show that under the assumptions of Theorem 1.1, taking for initial distribution the invariant measure π_n° of J_n°, the conditions of Theorem 8.1 are satisfied $\mathbb{P}$-almost surely. Firstly, by Theorem 8.2 and Proposition 8.5, Conditions (C1) and (C2) are satisfied $\mathbb{P}$-almost surely. That is, $\mathbb{P}$-almost surely the following holds: for all $u > 0$ and all $t > 0$,

$$\lim_{n\to\infty} \nu_n^{J_n^\circ,t}(u,\infty) = t\nu^\circ(u,\infty) \quad \text{in } \mathcal{P}^\circ\text{-probability}, \tag{8.71}$$

$$\lim_{n\to\infty} \sigma_n^{J_n^\circ,t}(u,\infty) = 0 \quad \text{in } \mathcal{P}^\circ\text{-probability}. \tag{8.72}$$

Next, in view of (8.15), (8.5) reads $\nu_n^\circ(v,\infty)/a_n = o(1)$, and so, by (8.52) of Proposition 8.5, Condition (C0) is satisfied. It remains to check Condition (C3). As in the proof of Proposition 8.5, we do this by comparing the quantity

$$\begin{aligned} \eta_n^\circ(\epsilon) &\equiv \lfloor a_n \rfloor \mathcal{E}_{\pi_n^\circ}^\circ \mathbb{1}_{\{\lambda_n^{-1}(J_n^\circ(0))e_0^\circ \leq c_n\epsilon\}} c_n^{-1}\lambda_n^{-1}(J_n^\circ(0))e_0^\circ && (8.73)\\ &= \frac{\lfloor a_n \rfloor}{|\mathcal{V}_n^\circ|} \sum_{x\in\mathcal{V}_n^\circ} c_n^{-1}\lambda_n^{-1}(x)\left(1 - e^{-\epsilon c_n \lambda_n(x)}\right) && (8.74) \end{aligned}$$

arising in (8.8), to its counterpart in the random hopping dynamics of the non truncated REM, $\eta_n^{\mathsf{REM}}(\epsilon)$, defined in (8.57). For this we simply write that since $\lambda_n(x) = 1$ on $\mathcal{V}_n^\circ \setminus I_n^\star$ and $c_n\lambda_n(x) = 1/\gamma_n(x)$ on $I_n^\star$,

$$\eta_n^\circ(\epsilon) \leq \frac{\lfloor a_n \rfloor}{c_n} + \frac{\lfloor a_n \rfloor}{|\mathcal{V}_n^\circ|} \sum_{x\in I_n^\star} \gamma_n(x)\left(1 - e^{-\epsilon/\gamma_n(x)}\right) \leq \frac{\lfloor a_n \rfloor}{c_n} + \frac{|\mathcal{V}_n|}{|\mathcal{V}_n^\circ|}\eta_n^{\mathsf{REM}}(\epsilon). \tag{8.75}$$

From this, (6.7), and (8.60) it follows that, under the assumptions of Proposition 8.6,

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} \eta_n^\circ(\epsilon) = 0, \qquad \mathbb{P}\text{-almost surely}. \tag{8.76}$$

Therefore Condition (C3) is satisfied $\mathbb{P}$-almost surely.

Since all four conditions (C0), (C1), (C2), and (C3) are satisfied $\mathbb{P}$-almost surely, it follows from Theorem 8.1 that, for our choices of a_n, c_n, β, and $c_\star$, $\mathbb{P}$-almost surely,

$$S_n^\circ \Rightarrow_{J_1} S_\infty^\circ \tag{8.77}$$

where S_∞° is a subordinator with zero drift and Lévy measure $\nu^\circ = \nu$ defined in (1.21). The proof of Theorem 3.2 is complete.

9. CONVERGENCE OF THE BACK END CLOCK PROCESS BELOW THE CRITICAL TEMPERATURE: PROOF OF THEOREM 3.4

9.1. A convergence theorem for BECP.

Consider the rescaled process (3.16),

$$S_n^\dagger(t) = b_n^{-1}\widetilde{S}_n^\dagger(k_n^\dagger(t)), \quad t \geq 0. \tag{9.1}$$

Theorem 9.1 below parallels Theorem 8.1 for FECP, namely, it gives three sufficient conditions for the sequence $S_n^\dagger$ to converge to a subordinator when the initial distribution of $J_n^\dagger$ is the invariant measure π_n° of J_n°. As before these conditions refer to given sequences of numbers a_n and b_n, and a given realization of the random environment. For $u > 0$ define

$$\bar{h}_n^u(y) = \sum_{1\leq l\leq L^\star} \sum_{x\in C_{n,l}^\star} p_n(y,x)P_x(T_{n,l}^\star > b_n u)\,,\ y \in \mathcal{V}_n^\circ, \tag{9.2}$$

where $T_{n,l}^\star$ is the exit time (5.6). (Note that $\bar{h}_n^u(y) = 0$ unless $y \in \cup_{1\leq l\leq L^\star}\partial C_{n,l}^\star$.) For $k_n^\circ(t)$ as in (3.19) define, for $t > 0$ and $u > 0$,

$$\bar{\nu}_n^{J_n^\circ,t}(u,\infty) = \sum_{j=0}^{k_n^\circ(t)-1} \bar{h}_n^u(J_n^\circ(j)), \tag{9.3}$$

$$\bar{\sigma}_n^{J_n^\circ,t}(u,\infty) = \sum_{j=0}^{k_n^\circ(t)-1} \left[\bar{h}_n^u(J_n^\circ(j))\right]^2. \tag{9.4}$$

Condition (A1). There exists a σ-finite measure $\nu^\dagger$ on $(0,\infty)$ satisfying $\int_0^\infty (1\wedge u)\nu^\dagger(du) < \infty$ such that, for all $t > 0$ and all $u > 0$,

$$P_{\pi_n^\circ}^\circ\left(\left|\bar{\nu}_n^{J_n^\circ,t}(u,\infty) - t\nu^\dagger(u,\infty)\right| < \epsilon\right) = 1 - o(1)\,, \quad \forall\epsilon > 0\,. \tag{9.5}$$

Condition (A2). For all $u > 0$ and all $t > 0$,

$$P_{\pi_n^\circ}^\circ\left(\bar{\sigma}_n^{J_n^\circ,t}(u,\infty) < \epsilon\right) = 1 - o(1)\,, \quad \forall\epsilon > 0\,. \tag{9.6}$$

Condition (A3). For all $t > 0$,

$$\lim_{\epsilon\downarrow 0}\limsup_{n\uparrow\infty} \frac{k_n^\circ(t)}{|\mathcal{V}_n^\circ|} \sum_{1\leq l\leq L^\star} \sum_{x\in C_{n,l}^\star} E_x\left(\mathbb{1}_{\{b_n^{-1}T_{n,l}^\star\leq\epsilon\}} b_n^{-1}T_{n,l}^\star\right) = 0. \tag{9.7}$$

Theorem 9.1. *Choose for initial distribution the invariant measure π_n° of J_n°. For all sequences a_n and b_n for which Conditions (A1), (A2), and (A3) are verified $\mathbb{P}$-almost surely,*

$$S_n^\dagger \Rightarrow_{J_1} S_\infty^\dagger \tag{9.8}$$

$\mathbb{P}$-almost surely, where $S_\infty^\dagger$ is the Lévy subordinator with zero drift and Lévy measure $\nu^\dagger$.

Proof of Theorem 9.1. The proof of Theorem 3.4 relies on Theorem 2.1 of [26], which is itself a specialization of Theorem 4.1 of [22] to processes with non-negative increments. Throughout we fix a realisation $\omega \in \Omega^\star$ of the random environment but do not make this explicit in the notation. With the notations of Subsection 3.2 define, for $i \geq 0$,

$$Z_{n,i} \equiv b_n^{-1}\Lambda_n^\dagger(i)\,. \tag{9.9}$$

Thus, by (3.11) and (9.1),

$$S_n^\dagger(t) = \textstyle\sum_{i=0}^{k_n^\dagger(t)-1} Z_{n,i}. \tag{9.10}$$

In view of (3.15), $k_n^\dagger(t)$ is a stopping time for each $t > 0$. Furthermore, because $J_n^\dagger$ starts in π_n°, and because $\pi_n^\circ(\mathcal{V}_n \setminus \mathcal{V}_n^\circ) = 0$, it follows from (3.12) that $Z_{n,0} = 0$. We may thus apply Theorem 2.1 of [26] to the sum (9.10).

To this end let $\{\mathcal{F}_{n,i}^\dagger, n \geq 1, i \geq 0\}$ be the array of sub-sigma fields defined (with obvious notation) by $\mathcal{F}_{n,i}^\dagger = \sigma\left(J_n^\dagger(0), \ldots, J_n^\dagger(i)\right)$, for $i \geq 0$. Clearly, for each n and $i \geq 1$, $Z_{n,i}$ is $\mathcal{F}_{n,i}^\dagger$ measurable and $\mathcal{F}_{n,i-1}^\dagger \subset \mathcal{F}_{n,i}^\dagger$. Next, observe that

$$\begin{aligned}\mathcal{P}_{\pi_n^\circ}^\dagger\big(Z_{n,i} > u \,\big|\, \mathcal{F}_{n,i-1}^\dagger\big) &= \textstyle\sum_{x\in\mathcal{V}_n} \mathcal{P}_{\pi_n^\circ}^\dagger\big(J_n^\dagger(i) = x, Z_{n,i} > u \,\big|\, J_n^\dagger(i-1)\big)\\ &= \textstyle\sum_{x\in\mathcal{V}_n} \mathcal{P}_{\pi_n^\circ}^\dagger\big(J_n^\dagger(i) = x, \Lambda_n^\dagger(i) > b_n u \,\big|\, J_n^\dagger(i-1)\big). \end{aligned}\tag{9.11}$$

By (3.12) and (5.6), $\Lambda_n^\dagger(i) = 0$ if $x \notin \cup_{1\leq l\leq L^\star} C_{n,l}^\star$, and $\Lambda_n^\dagger(i) = T_{n,l}^\star$ if $x \in C_{n,l}^\star$. Thus,

$$\mathcal{P}_{\pi_n^\circ}^\dagger\big(Z_{n,i} > u \,\big|\, \mathcal{F}_{n,i-1}^\dagger\big) = \sum_{1\leq l\leq L^\star}\sum_{x\in C_{n,l}^\star} \mathcal{P}_{\pi_n^\circ}^\dagger\big(J_n^\dagger(i) = x, \Lambda_n^\dagger(i) > b_n u \,\big|\, J_n^\dagger(i-1)\big). \tag{9.12}$$

Now for all $1 \leq l \leq L^\star$ and all $x \in C_{n,l}^\star$,

$$\mathcal{P}_{\pi_n^\circ}^\dagger\big(J_n^\dagger(i) = x, \Lambda_n^\dagger(i) > b_n u \,\big|\, J_n^\dagger(i-1)\big) = p_n^\dagger\big(J_n^\dagger(i-1), x\big) P_x\big(T_{n,l}^\star > b_n u\big) \tag{9.13}$$

where, by (3.13),

$$p_n^\dagger\big(J_n^\dagger(i-1), x\big) = p_n\big(J_n^\dagger(i-1), x\big)\mathbb{1}_{\{J_n^\dagger(i-1)\in\mathcal{V}_n^\circ\}} \tag{9.14}$$

(indeed, by definition of $J_n^\dagger$, $J_n^\dagger(i) \in C_{n,l}^\star$ if and only if $J_n^\dagger(i-1) \in \partial C_{n,l}^\star \subset \mathcal{V}_n^\circ$). In view of (9.2) it follows from (9.11), (9.12), (9.13) and (9.14) that

$$\textstyle\sum_{i=1}^{k_n^\dagger(t)} \mathcal{P}_{\pi_n^\circ}^\dagger\big(Z_{n,i} > u \,\big|\, \mathcal{F}_{n,i-1}^\dagger\big) = \sum_{i=1}^{k_n^\dagger(t)} \bar{h}_n^u\big(J_n^\dagger(i-1)\big)\mathbb{1}_{\{J_n^\dagger(i-1)\in\mathcal{V}_n^\circ\}}. \tag{9.15}$$

It remains to notice that the chain $J_n^\dagger$ observed only when it takes values in $\mathcal{V}_n^\circ$ is nothing but the chain J_n°, and that J_n° takes $k_n^\circ(t)$ steps when $J_n^\dagger$ takes $k_n^\dagger(t)$ steps (see (3.19)). Thus,

$$\textstyle\sum_{i=1}^{k_n^\dagger(t)} \bar{h}_n^u\big(J_n^\dagger(i-1)\big)\mathbb{1}_{\{J_n^\dagger(i-1)\in\mathcal{V}_n^\circ\}} \stackrel{d}{=} \sum_{i=1}^{k_n^\circ(t)} \bar{h}_n^u\big(J_n^\circ(i-1)\big) = \bar{\nu}_n^{J_n^\circ,t}(u,\infty), \tag{9.16}$$

where the first equality holds in distribution and the last is (9.3). Combining (9.15) and (9.16) now yields

$$\textstyle\sum_{i=1}^{k_n^\dagger(t)} \mathcal{P}_{\pi_n^\circ}^\dagger\big(Z_{n,i} > u \,\big|\, \mathcal{F}_{n,i-1}^\dagger\big) \stackrel{d}{=} \bar{\nu}_n^{J_n^\circ,t}(u,\infty). \tag{9.17}$$

Similarly, we get

$$\textstyle\sum_{i=1}^{k_n(t)-1} \left[\mathcal{P}_{\pi_n^\circ}^\dagger\big(Z_{n,i} > u \,\big|\, \mathcal{F}_{n,i-1}^\dagger\big)\right]^2 \stackrel{d}{=} \sum_{i=1}^{k_n^\dagger(t)} \left[\bar{h}_n^u\big(J_n^\circ(i-1)\big)\right]^2 = \bar{\sigma}_n^{J_n^\circ,t}(u,\infty). \tag{9.18}$$

From (9.17) and (9.18) it follows that Conditions (A2) and (A1) of Theorem 9.1 are exactly the Conditions (D1) and (D2) of Theorem 2.1 of [26]. To see that Condition (A3) implies Condition (D3) we have to establish that (9.7) implies

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} \mathcal{P}_{\pi_n^\circ}^\dagger\left(S_n^{\dagger,\epsilon}(t) > \epsilon\right) = 0\,, \tag{9.19}$$

where for $\epsilon \geq 0$, $S_n^{\dagger,\epsilon}(t) = \sum_{i=0}^{k_n^\dagger(t)-1} Z_{n,i}\mathbb{1}_{\{Z_{n,i}\leq\epsilon\}}$. By Theorem 3.1 with $c_\circ = 1$, on $\Omega^\star$, for all but a finite number of indices n, all $0 < t < \infty$, and all $\epsilon \geq 0$,

$$\mathcal{P}^\dagger_{\pi_n^\circ}\left(S_n^{\dagger,\epsilon}(t) > \epsilon\right) \leq \mathcal{P}^\dagger_{\pi_n^\circ}\left(\sum_{i=0}^{k_n(t)-1} Z_{n,i}\mathbb{1}_{\{Z_{n,i}\leq\epsilon\}} > \epsilon\right) + n^{-2(c_\star-1)+1}(1+o(1)), \quad (9.20)$$

where $k_n(t) \equiv \lfloor k_n^\circ(t)(1+n^{-1})\rfloor = \lfloor\lfloor a_n t\rfloor(1+n^{-1})\rfloor$. By Tchebychev inequality,

$$\mathcal{P}^\dagger_{\pi_n^\circ}\left(\sum_{i=0}^{k_n(t)-1} Z_{n,i}\mathbb{1}_{\{Z_{n,i}\leq\epsilon\}} > \epsilon\right) \leq \epsilon^{-1}\sum_{i=1}^{k_n(t)-1}\mathcal{E}^\dagger_{\pi_n^\circ}\mathbb{1}_{\{Z_i^n\leq\epsilon\}}Z_i^n, \quad (9.21)$$

and the right hand side of (9.21) is equal to

$$\sum_{1\leq l\leq L^\star}\sum_{x\in C^\star_{n,l}} E_x\left(\mathbb{1}_{\{b_n^{-1}T^\star_{n,l}\leq\epsilon\}}b_n^{-1}T^\star_{n,l}\right)\sum_{i=1}^{k_n(t)-1}\mathcal{E}^\dagger_{\pi_n^\circ}\mathbb{1}_{\{J_n^\dagger(i)=x\}}. \quad (9.22)$$

By (7.27) and (3.13), for all $x \in C^\star_{n,l}$,

$$\begin{aligned}\sum_{i=1}^{k_n(t)-1}\mathcal{E}^\dagger_{\pi_n^\circ}\mathbb{1}_{\{J_n^\dagger(i)=x\}} &= \sum_{y\in\partial C^\star_{n,l}\cap\partial x} p_n(y,x)\sum_{i=1}^{k_n(t)-1}\mathcal{E}^\dagger_{\pi_n^\circ}\mathbb{1}_{\{J_n^\dagger(i-1)=y\}}\\ &\leq \sum_{y\in\partial C^\star_{n,l}\cap\partial x} p_n(y,x)\sum_{i=1}^{k_n(t)-1}\pi_n^\circ(y)\\ &\leq (k_n(t)-1)/|\mathcal{V}_n^\circ| \end{aligned} \quad (9.23)$$

where the last inequality follows from (6.6) and the fact that, for $y \in \partial C^\star_{n,l}\cap\partial x$ and $x \in C^\star_{n,l}$, $p_n(y,x) = n^{-1}$. Combining (9.21), (9.22), and (9.23), the probability in the left hand side of (9.21) is bounded above by

$$\epsilon^{-1}(1+n^{-1})(k_n^\circ(t)/|\mathcal{V}_n^\circ|)\sum_{1\leq l\leq L^\star}\sum_{x\in C^\star_{n,l}} E_x\left(\mathbb{1}_{\{b_n^{-1}T^\star_{n,l}\leq\epsilon\}}b_n^{-1}T^\star_{n,l}\right). \quad (9.24)$$

Inserting this bound in (9.20) yields the claim that Condition (A3) implies Condition (D3) of Theorem 2.1 of [26]

Having established that, on $\Omega^\star$, the conditions of Theorem 2.1 of [26] are verified whenever those of Theorem 9.1 are verified, the proof of Theorem 9.1 is complete. □

9.2. **An ergodic theorem for BECP.** To prove that Conditions (A1) and (A2) of Theorem 9.1 are satisfied we closely follow the strategy of Subsection 8.2 and first prove an ergodic theorem for the quantities $\bar\nu_n^{J_n^\circ,t}(u,\infty)$ and $\bar\sigma_n^{J_n^\circ,t}(u,\infty)$ defined in (9.3) and (9.4). Clearly, for $\pi_n^{J_n^\circ,t}(x)$ as in (8.10), (9.3) and (9.4) can be rewritten as

$$\begin{aligned}\bar\nu_n^{J_n^\circ,t}(u,\infty) &= k_n^\circ(t)\sum_{y\in\mathcal{V}_n^\circ}\pi_n^{J_n^\circ,t}(y)\bar h_n^u(y), &(9.25)\\ \bar\sigma_n^{J_n^\circ,t}(u,\infty) &= k_n^\circ(t)\sum_{y\in\mathcal{V}_n^\circ}\pi_n^{J_n^\circ,t}(y)\left[\bar h_n^u(y)\right]^2. &(9.26)\end{aligned}$$

Before stating our main theorem, let us express the mean values of (9.25) and (9.26) with respect to the law $P^\circ_{\pi_n^\circ}$. Given $x \in C^\star_{n,l}$, $1 \leq l \leq L^\star$, denote by

$$Q^u_{n,l}(x) \equiv P_x(T^\star_{n,l} > b_n u), \quad u > 0, \quad (9.27)$$

the tail distribution of the exit time $T^\star_{n,l}$ given that the set $C^\star_{n,l}$ is entered in x, and define

$$\bar\nu_n^\circ(u,\infty) = \frac{a_n}{2^n}\sum_{1\leq l\leq L^\star}\sum_{x\in C^\star_{n,l}} Q^u_{n,l}(x), \quad (9.28)$$

$$\bar\sigma_n^=(u,\infty) = \frac{a_n}{n2^n}\sum_{1\leq l\leq L^\star}\Big[\sum_{x\in C^\star_{n,l}} Q^u_{n,l}(x)\Big]^2, \quad (9.29)$$

$$\bar{\sigma}_n^\circ(u,\infty)=\frac{a_n}{2^n}\sum_{1\le l\le L^\star}\sum_{1\le l'\le L^\star}\sum_{x\in C^\star_{n,l}}\sum_{x'\in C^\star_{n,l'}}Q^u_{n,l}(x)Q^u_{n,l'}(x')\left(n^{-2}|\partial x\cap\partial x'|\right). \tag{9.30}$$

Lemma 9.2. *Assume that $c_\star>2$. Then on $\Omega^\star$, for all but a finite number of indices n,*

$$E^\circ_{\pi^\circ_n}\left[\bar{\nu}_n^{J^\circ_n,t}(u,\infty)\right]=(1+o(1))(k^\circ_n(t)/a_n)\bar{\nu}^\circ_n(u,\infty), \tag{9.31}$$

$$E^\circ_{\pi^\circ_n}\left[\bar{\sigma}_n^{J^\circ_n,t}(u,\infty)\right]=(1+o(1))(k^\circ_n(t)/a_n)\bar{\sigma}^\circ_n(u,\infty). \tag{9.32}$$

The main theorem of this section controls the fluctuations of $\bar{\nu}_n^{J^\circ_n,t}$ around its mean value and provides an upper bound on $\bar{\sigma}_n^{J^\circ_n,t}$ in terms of the random (in the environment) quantities $\bar{\nu}^\circ_n$, $\bar{\sigma}^\circ_n$, and $\bar{\sigma}^=_n$.

Theorem 9.3. *Assume that $c_\star>3$. Let $\bar{\rho}^\circ_n>0$ be a decreasing sequence satisfying $\bar{\rho}^\circ_n\downarrow 0$ as $n\uparrow\infty$. There exists a sequence of subsets $\overline{\Omega}^{EG}_n\subset\Omega$ with $\mathbb{P}\left[\left(\overline{\Omega}^{EG}_n\right)^c\right]<2^6\ell^\circ_n/(\bar{\rho}^\circ_n n a_n)+n^{-2}$, and such that on $\overline{\Omega}^{EG}_n$ the following holds for all large enough n: for all $t>0$, all $u>0$, and all $\epsilon>0$,*

$$P^\circ_{\pi^\circ_n}\left(\left|\bar{\nu}_n^{J^\circ_n,t}(u,\infty)-E^\circ_{\pi^\circ_n}\left[\bar{\nu}_n^{J^\circ_n,t}(u,\infty)\right]\right|\ge\epsilon\right)\le\epsilon^{-2}[C_3t\Theta_{n,3}(u)+t^2\Theta_{n,4}(u)] \tag{9.33}$$

for some constant $0<C_3<\infty$ and where, for $\varsigma^{\neq}_n(u)$ as in Lemma 9.5,

$$\Theta_{n,3}(u)\equiv\bar{\sigma}^\circ_n(u,\infty)+\bar{\sigma}^=_n(u,\infty)+\frac{1}{\log n}\mathbb{E}\bar{\sigma}^=_n(u,\infty)+\bar{\rho}^\circ_n\left[\varsigma^{\neq}_n(u)\right]^2, \tag{9.34}$$

$$\Theta_{n,4}(u)\equiv 2^{-n}\left[\bar{\nu}^\circ_n(u,\infty)\right]^2. \tag{9.35}$$

Moreover, for all $t>0$, all $u>0$, and all $\epsilon'>0$,

$$P^\circ_{\pi^\circ_n}\left(\bar{\sigma}_n^{J^\circ_n,t}(u,\infty)\ge\epsilon'\right)\le\frac{t}{\epsilon'}(1+o(1))\bar{\sigma}^\circ_n(u,\infty). \tag{9.36}$$

We now prove, in this order, Lemma 9.2 and Theorem 9.3.

Proof of Lemma 9.2. By (9.2), (9.25), and (9.27),

$$E^\circ_{\pi^\circ_n}\left[\bar{\nu}_n^{J^\circ_n,t}(u,\infty)\right]=k^\circ_n(t)\textstyle\sum_{y\in\mathcal{V}^\circ_n}\pi^\circ_n(y)\bar{h}^u_n(y) \tag{9.37}$$

$$=k^\circ_n(t)\textstyle\sum_{y\in\mathcal{V}^\circ_n}\pi^\circ_n(y)\sum_{1\le l\le L^\star}\sum_{x\in C^\star_{n,l}}p_n(y,x)Q^u_{n,l}(x), \tag{9.38}$$

and since both x and y belong to $\mathcal{V}^\circ_n$, $p_n(y,x)=n^{-1}$ if $\mathrm{dist}(x,y)=1$ and is zero else. Thus $\sum_{y\in\mathcal{V}^\circ_n}p_n(y,x)=n^{-1}|\partial x\cap\partial C^\star_{n,l}|$ and

$$E^\circ_{\pi^\circ_n}\left[\bar{\nu}_n^{J^\circ_n,t}(u,\infty)\right]=(k^\circ_n(t)/|\mathcal{V}^\circ_n|)\textstyle\sum_{1\le l\le L^\star}\sum_{x\in C^\star_{n,l}}n^{-1}|\partial x\cap\partial C^\star_{n,l}|Q^u_{n,l}(x). \tag{9.39}$$

The claim of (9.31) now follows from (2.14) and (6.7). Eq. (9.32) is proved in a similar way. We skip the details. □

Proof of Theorem 9.3 . A first order Tchebychev inequality and (9.32) readily yield (9.36). As in Theorem 8.2, proving concentration of $\bar{\nu}_n^{J^\circ_n,t}(u,\infty)$ is more involved. Since (9.25) is nothing but (8.11) with h^u_n replaced by $\bar{h}^u_n$, the proof naturally starts in the same way as the proof of (8.18) of Theorem 8.2. More precisely, substituting $\bar{h}^u_n$ for h^u_n in the definition (8.29) of the quantities $I^{(2)}_i$, $1\le i\le 3$, we get that for all $\epsilon>0$,

$$P^\circ_{\pi^\circ_n}\left(\left|\bar{\nu}_n^{J^\circ_n,t}(u,\infty)-E^\circ_{\pi^\circ_n}\left[\bar{\nu}_n^{J^\circ_n,t}(u,\infty)\right]\right|\ge\epsilon\right)\le\epsilon^{-2}[I^{(2)}_1+I^{(2)}_2+I^{(2)}_3]. \tag{9.40}$$

We are thus left to bound $I^{(2)}_i$, $1\le i\le 3$. By (9.25) and (9.31),

$$I^{(2)}_1=2^{-n}\left[E^\circ_{\pi^\circ_n}\left[\bar{\nu}_n^{J^\circ_n,t}(u,\infty)\right]\right]^2\le 2^{-n}\left(k^\circ_n(t)/a_n\right)^2\left[\bar{\nu}^\circ_n(u,\infty)\right]^2, \tag{9.41}$$

and by (9.26) and (9.32),

$$I_2^{(2)} \leq E^{\circ}_{\pi_n^{\circ}}\left[\bar{\sigma}_n^{J_n^{\circ},t}(u,\infty)\right] \leq (k_n^{\circ}(t)/a_n)\bar{\sigma}_n^{\circ}(u,\infty). \tag{9.42}$$

The term $I_3^{(2)}$ is a little more involved. We may write it in the form

$$I_3^{(2)} \equiv 2(k_n^{\circ}(t)/a_n)(a_n/|\mathcal{V}_n^{\circ}|)\sum_{m=1}^{\ell_n^{\circ}-1}[I_{0,m}^{(3)} + I_{1,m}^{(3)} + I_{2,m}^{(3)}], \tag{9.43}$$

where, setting $f_n^{\circ,m}(x,x';y,y') \equiv p_n(y,x)p_n(y',x')p_n^{\circ,m}(y,y')$, and for $\mathcal{W}_n$ given in (6.3),

$$I_{0,m}^{(3)} \equiv \sum_{1\leq l\leq L^{\star}}\sum_{x\in C_{n,l}^{\star}}\sum_{x'\in C_{n,l}^{\star}}\sum_{y\in\partial C_{n,l}^{\star}}\sum_{y'\in\partial C_{n,l}^{\star}\cap\mathcal{W}_n} Q_{n,l}^u(x)Q_{n,l}^u(x')f_n^{\circ,m}(x,x';y,y'), \tag{9.44}$$

$$I_{1,m}^{(3)} \equiv \sum_{1\leq l\leq L^{\star}}\sum_{x\in C_{n,l}^{\star}}\sum_{x'\in C_{n,l}^{\star}}\sum_{y\in\partial C_{n,l}^{\star}}\sum_{y'\in\partial C_{n,l}^{\star}\cap\mathcal{W}_n^c} Q_{n,l}^u(x)Q_{n,l}^u(x')f_n^{\circ,m}(x,x';y,y'), \tag{9.45}$$

$$I_{2,m}^{(3)} \equiv \sum_{1\leq l,l'\leq L^{\star}:l\neq l'}\sum_{x\in C_{n,l}^{\star}}\sum_{x'\in C_{n,l'}^{\star}}\sum_{y\in\partial C_{n,l}^{\star}}\sum_{y'\in\partial C_{n,l'}^{\star}} Q_{n,l}^u(x)Q_{n,l'}^u(x')f_n^{\circ,m}(x,x';y,y'). \tag{9.46}$$

Consider $I_{0,m}^{(3)}$ first. It follows from Proposition 6.2, (ii) that on Ω^{SRW}, for all but a finite number of indices n, for all $x,x'\in C_{n,l}^{\star}$,

$$\sum_{m=1}^{\ell_n^{\circ}-1}\sum_{y\in\partial C_{n,l}^{\star}}\sum_{y'\in\partial C_{n,l}^{\star}} f_n^{\circ,m}(x,x';y,y') \leq \frac{C'_{\circ}}{\log n}\sum_{y\in\partial C_{n,l}^{\star}}\sum_{y'\in\partial C_{n,l}^{\star}} p_n(y,x)p_n(y',x') \leq \frac{C'_{\circ}}{\log n}. \tag{9.47}$$

(Here we used that $p_n(y,x) = p_n(y,x)$ if both x and y belong to $\mathcal{V}_n^{\circ}$.) From this and Proposition 6.3 we readily get that if $c_{\star} > 2$ then on $\Omega^{\mathsf{SRW}}\cap\Omega^{\star}$, for large enough n,

$$(a_n/|\mathcal{V}_n^{\circ}|)\textstyle\sum_{m=1}^{\ell_n^{\circ}-1} I_{0,m}^{(3)} \leq C'_{\circ}(1+o(1))\bar{\sigma}_n^{=}(u,\infty). \tag{9.48}$$

The next two lemmata bound the contribution to (9.43) coming from $I_{1,m}^{(3)}$ and $I_{2,m}^{(3)}$.

Lemma 9.4. *There exists a sequence of subsets* $\overline{\Omega}_{1,n}^{(3)} \subset \Omega$ *with* $\mathbb{P}\big(\overline{\Omega}_{1,n}^{(3)}\big) \geq 1 - n^{-2}$ *such that on* $\overline{\Omega}_{1,n}^{(3)}\cap\Omega^{\star}$, *if* $\kappa_{\star} = [(c_{\star}+2)/(c_{\star}-1)]^2$, $\sum_{m=1}^{\ell_n^{\circ}-1} I_{1,m}^{(3)} < \mathbb{E}\bar{\sigma}_n^{=}(u,\infty)/\log n$.

Proof. As is Lemma 8.3, this follows from a first order Tchebychev inequality and (2.16), using that $Q_{n,l}^u(x')$ is independent from the variables in $\partial y'\cap\partial\mathcal{V}_n^{\star}$ and $\partial y_2'\cap\partial\mathcal{V}_n^{\star}$. □

Lemma 9.5. *Assume that* $c_{\star} > 2$. *Let* $\bar{\rho}_n^{\circ} > 0$ *be a decreasing sequence satisfying* $\bar{\rho}_n^{\circ}\downarrow 0$ *as* $n\uparrow\infty$.*There exists a sequence of subsets* $\overline{\Omega}_{2,n}^{(3)} \subset \Omega$ *with* $\mathbb{P}\big(\overline{\Omega}_{2,n}^{(3)}\big) \geq 1 - 2^6\ell_n^{\circ}/(\bar{\rho}_n^{\circ} n a_n)$ *such that on* $\overline{\Omega}_{2,n}^{(3)}\cap\Omega^{\star}$, *for all* n *large enough,*

$$(a_n/|\mathcal{V}_n^{\circ}|)\textstyle\sum_{m=1}^{\ell_n^{\circ}-1} I_{2,m}^{(3)} < \bar{\rho}_n^{\circ}\left[\varsigma_n^{\neq}(u)\right]^2. \tag{9.49}$$

where $\varsigma_n^{\neq}(u)$ *is a positive decreasing function of* $u > 0$ *that satisfies*

$$\lim_{n\to\infty}\varsigma_n^{\neq}(u) = \nu^{\dagger}(u,\infty), \quad \forall u > 0. \tag{9.50}$$

The proof of lemma 9.5 is given in Subsection 9.3.2.

Equipped with 9.48 and Lemma 9.5 we conclude that under the assumptions and with the notations of Proposition 6.2, Lemma 9.5, and Proposition 6.3, on $\Omega^{\mathsf{SRW}}\cap\overline{\Omega}^{(3)}_{1,n}\cap\overline{\Omega}^{(3)}_{2,n}\cap\Omega^{\star}$, for all but a finite number of indices n,

$$I_3^{(2)} \leq 2\frac{k_n^{\circ}(t)}{a_n}\left(C_{\circ}''\bar{\sigma}_n^{=}(u,\infty) + \frac{1}{\log n}\mathbb{E}\bar{\sigma}_n^{=}(u,\infty) + \bar{\rho}_n^{\circ}\left[\varsigma_n^{\neq}(u)\right]^2\right) \tag{9.51}$$

for some constant $0 < C_{\circ}'' < \infty$. Inserting the bounds (9.41), (9.42), and (9.51) in (9.40) then yields (9.33)-(9.35). The proof of Theorem 9.3 is done. ☐

9.3. **Almost sure convergence of $\bar{\nu}_n^{\circ}$, $\bar{\sigma}_n^{\circ}$, and $\bar{\sigma}_n^{=}$.** As in Subsection 8.3 our next step consists in proving strong laws of large numbers for the random (but now chain independent) quantities $\bar{\nu}_n^{\circ}$, $\bar{\sigma}_n^{\circ}$, and $\bar{\sigma}_n^{=}$ defined in (9.28), (9.30), and (9.29), respectively. However, the complexity of these objects (note in particular that they are sums of correlated random variables) makes this task much more arduous than in FECP.

Proposition 9.6. *Given $0 < \varepsilon < 1$ let the sequences a_n and b_n be defined through*

$$\lim_{n\to\infty}\frac{\log a_n}{n\log 2} = \varepsilon, \quad \sqrt{na_n}\mathbb{P}(w_n(x) \geq (n-1)b_n) \sim 1, \tag{9.52}$$

Assume that $c_\star > 2$ and let $\nu^\dagger$ be as in (1.23). Then, there exists a subset $\overline{\Omega}^{LLN} \subset \Omega$ with $\mathbb{P}(\overline{\Omega}^{LLN}) = 1$ such that, on $\overline{\Omega}^{LLN}$, the following holds: for all $u > 0$,

$$\lim_{n\to\infty}\bar{\nu}_n^{\circ}(u,\infty) = \nu^{\dagger}(u,\infty), \tag{9.53}$$

$$\lim_{n\to\infty} n\bar{\sigma}_n^{\circ}(u,\infty) = \lim_{n\to\infty} n\bar{\sigma}_n^{=}(u,\infty) = 2\nu^{\dagger}(2u,\infty). \tag{9.54}$$

To prove Proposition 9.6 we first establish control over the mean values of $\bar{\nu}_n^{\circ}(u,\infty)$, $\bar{\sigma}_n^{\circ}(u,\infty)$, and $\bar{\sigma}_n^{=}(u,\infty)$ (see Lemmata 9.8 and 9.10), and then prove that these quantities concentrate around their means (in Lemmata 9.9, 9.11). Both these steps rely on the following key lemma. Given sequences $\bar{a}_n$, $\bar{b}_n$, and two distinct vertices $x, y \in \mathcal{V}_n$, set

$$v_n(u;\bar{a}_n,\bar{b}_n) = \bar{a}_n^2\mathbb{E}\left[\exp\left(-\tfrac{u\bar{b}_n}{\min\{w_n(x),w_n(y)\}}\right)\mathbb{1}_{\min\{w_n(x),w_n(y)\}\geq r_n(\rho_n^{\star})\}}\right]. \tag{9.55}$$

Lemma 9.7. *If the sequences $\bar{a}_n$ and $\bar{b}_n$ satisfy $\bar{a}_n\mathbb{P}(w_n(x) \geq \bar{b}_n) \sim 1$, $\lim_{n\to\infty}\frac{\bar{a}_n}{2^n} = 0$, and $\lim_{n\to\infty}\frac{\log\bar{a}_n}{n\log 2} = \bar{\varepsilon}$ for some $\bar{\varepsilon} > 0$, then*

$$\lim_{n\to\infty} v_n(u;\bar{a}_n,\bar{b}_n) = u^{-2\alpha(\bar{\varepsilon})}2\alpha(\bar{\varepsilon})\Gamma(2\alpha(\bar{\varepsilon})). \tag{9.56}$$

Proof. The Proof of Lemma 9.7 is analogous to that of Lemma 5.3 of [25]. Details can be found in the arxiv extended version of the present work. ☐

The rest of Subsection 9.3 is organized as follows. The convergence properties of $\bar{\nu}_n^{\circ}(u,\infty)$ are established in Subsection 9.3.1 and those of $\bar{\sigma}_n^{\circ}(u,\infty)$ and $\bar{\sigma}_n^{=}(u,\infty)$ in Subsection 9.3.2. Subsection 9.3.2 also contains the proof of of Lemma 9.5. The proof of Proposition 9.6 is then completed in Subsection 9.3.3.

9.3.1. *Convergence properties of $\bar{\nu}_n^{\circ}$.* As stated in the next two lemmata $\bar{\nu}_n^{\circ}$ concentrates around its mean, and the mean as a limit.

Lemma 9.8. *Assume that $c_\star > 2$. If a_n and b_n satisfy (9.52) for some $0 < \varepsilon < 1$, then*

$$\lim_{n\to\infty}\mathbb{E}[\bar{\nu}_n^{\circ}(u,\infty)] = \nu^{\dagger}(u,\infty), \quad \forall u > 0. \tag{9.57}$$

Lemma 9.9. *For all* $L_1 > 0$ *and* $L_2 \geq 0$ *such that* $na_nL_2/2^n = o(1)$*, for all* $u > 0$*,*

$$\mathbb{P}\big(|\bar{\nu}_n^\circ(u,\infty) - \mathbb{E}[\bar{\nu}_n^\circ(u,\infty)]| \geq \phi_n(u,L_1,L_2)\big) \leq 4n^5e^{-L_2} + 2L_1, \tag{9.58}$$

where $\phi_n(u,L_1,L_2) \equiv 4n^4\sqrt{na_nL_2V_{n,2}(2u)/2^n} + 4n^{-(c_\star-1)}V_{n,1}(u)/L_1$*, and where for each* n*,* $V_{n,1}(u)$ *and* $V_{n,2}(u)$ *are positive decreasing functions, while for each* $u > 0$*, under the assumptions of Lemma 9.8,*

$$\lim_{n\to\infty} V_{n,1}(u) = \lim_{n\to\infty} V_{n,2}(u) = \nu^\dagger(u,\infty). \tag{9.59}$$

Proof of Lemma 9.8. Write $\bar{\nu}_n^\circ(u,\infty) = \sum_{k\geq 2}\bar{\nu}_n^{\circ,(k)}(u,\infty)$ where

$$\bar{\nu}_n^{\circ,(k)}(u,\infty) \equiv \frac{a_n}{2^n}\sum_{1\leq l\leq L^\star}\mathbb{1}_{\{|C^\star_{n,l}|=k\}}\sum_{x\in C^\star_{n,l}}Q^u_{n,l}(x). \tag{9.60}$$

We saw in Subsection 7.1 (see (7.7)) that on $\Omega^\star$, $k_n^\star \equiv \max_{2\leq l\leq L^\star}|C^\star_{n,l}(\rho)| \leq \frac{n}{(c_\star-2)\log n}$ for all large enough n. We may thus restrict the range of k to $2 \leq k \leq k_n^\star$. Let now $\mathcal{G}_k$ be the collection of all vertex sets $\mathcal{C} \subset \mathcal{V}_n$ of size k such that $G(\mathcal{C})$ forms a connected subgraph of $G(\mathcal{V}_n)$,

$$\mathcal{G}_k = \{\mathcal{C} = \{x_1,\dots,x_k\} : \forall_{1\leq i\leq k}\exists_{1\leq j\leq k} \text{ such that } \operatorname{dist}(x_i,x_j) = 1\}. \tag{9.61}$$

Then (9.60) can be written as

$$\bar{\nu}_n^{\circ,(k)}(u,\infty) \equiv \frac{a_n}{2^n}\sum_{\mathcal{C}\in\mathcal{G}_k}\prod_{x\in\mathcal{C}}\chi_n(x)\prod_{x'\in\partial\mathcal{C}}\overline{\chi}_n(x')\sum_{x\in\mathcal{C}}Q^u_{n,\mathcal{C}}(x), \tag{9.62}$$

where $Q^u_{n,\mathcal{C}}(x)$ stands for $Q^u_{n,l}(x)$ with $C^\star_{n,l} \equiv \mathcal{C}$, and where $\chi_n(x) \equiv \mathbb{1}_{\{w_n(x)\geq r_n(\rho_n^\star)\}}$, $\overline{\chi}_n(x) \equiv 1-\chi_n(x)$, are Bernoulli variables r.v.'s with $\mathbb{P}(\chi_n(x)=1) = n^{-c_\star}$. To further express $\bar{\nu}_n^{\circ,(k)}$ we distinguish the case $k=2$ from the case $3\leq k\leq k_n^\star$.

• ***The case*** $k=2$*.* Here $\mathcal{G}_2$ is the set of undirected edges of $G(\mathcal{V}_n)$ and $Q^u_{n,\mathcal{C}}(x)$ is given by Proposition 5.1, (i) : observing that $Q^u_{n,\mathcal{C}}(x) = Q^u_{n,\mathcal{C}}(y)$ on $\mathcal{C} = \{x,y\}$, and that, by (5.1), $\varrho_{n,l}(0) = \min_{x\in\mathcal{C}}w_n(x)$ when $C^\star_{n,l} = \mathcal{C}$, we obtain

$$\bar{\nu}_n^{\circ,(2)}(u,\infty) \equiv 2\frac{a_n}{2^n}\sum_{\mathcal{C}\in\mathcal{G}_2}\prod_{x\in\mathcal{C}}\chi_n(x)\prod_{x'\in\partial\mathcal{C}}\overline{\chi}_n(x')\left(1-\frac{1}{1+\frac{\min_{x\in\mathcal{C}}w_n(x)}{(n-1)}}\right)^{\lceil b_nu\rceil}. \tag{9.63}$$

From this we easily derive the bounds

$$\bar{\nu}_n^{\circ,(2),-}(u,\infty)(1-s_n) \leq \bar{\nu}_n^{\circ,(2)}(u,\infty) \leq \bar{\nu}_n^{\circ,(2),+}(u,\infty) \tag{9.64}$$

where $s_n = \frac{n-1}{r_n(\rho_n^\star)}$ and where, setting $b_n^\pm = b_n(n-1)(1-s_n)^{\pm1}$ and

$$\gamma_n^\pm(\mathcal{C}) = \min_{x\in\mathcal{C}} w_n(x)/b_n^\pm, \tag{9.65}$$

$$\bar{\nu}_n^{\circ,(2),\pm}(u,\infty) \equiv 2\frac{a_n}{2^n}\sum_{\mathcal{C}\in\mathcal{G}_2}\prod_{x'\in\partial\mathcal{C}}\overline{\chi}_n(x')e^{-u/\gamma_n^\pm(\mathcal{C})}\mathbb{1}_{\{\gamma_n^\pm(\mathcal{C})\geq r_n(\rho_n^\star)/b_n^\pm\}}. \tag{9.66}$$

By Lemma 9.7,

$$\lim_{n\to\infty}\mathbb{E}\bar{\nu}_n^{\circ,(2),-}(u,\infty) = \lim_{n\to\infty}\mathbb{E}\bar{\nu}_n^{\circ,(2),+}(u,\infty) = \nu^\dagger(u,\infty). \tag{9.67}$$

To see this note that, setting $a_n^+ = a_n^- = \sqrt{na_n}(1-n^{-c_\star})^{n-1}$,

$$\mathbb{E}[\bar{\nu}_n^{\circ,(2),\pm}(u,\infty)] = v_n(u;a_n^\pm,b_n^\pm). \tag{9.68}$$

One then readily checks that for a_n, b_n as in (9.52), $\lim_{n\to\infty} \frac{\log a_n^\pm}{n\log 2} = \varepsilon/2$, $\lim_{n\to\infty} \frac{a_n^\pm}{2^n} = 0$, and $a_n^\pm \mathbb{P}(w_n(x) \geq b_n^\pm) \sim 1$. The conditions of Lemma 9.7 are thus satisfied with $\bar\varepsilon = \varepsilon/2$, yielding (9.67). Since clearly $\lim_{n\to\infty} s_n = 0$, it follows from (9.64) that

$$\lim_{n\to\infty} \mathbb{E}\bar\nu_n^{\circ,(2)}(u,\infty) = \nu^\dagger(u,\infty). \tag{9.69}$$

• ***The case*** $3 \leq k \leq k_n^\star$**.** Recall that $Q^u_{n,\mathcal{C}}(x)$ in (9.62) stands for $Q^u_{n,l}(x)$ with $C^\star_{n,l} \equiv \mathcal{C}$. Similarly, denote by $\varrho_{n,\mathcal{C}}(0)$ the quantity $\varrho_{n,l}(0)$ from (5.1) with $C^\star_{n,l} \equiv \mathcal{C}$. By (5.3) of Proposition 5.1, (ii), on $\Omega^\star$, for all but a finite number of indices n,

$$Q^u_{n,\mathcal{C}}(x) \leq e^{-\{u(n-1)b_n/\varrho_{n,\mathcal{C}}(0)\}}(1+o(1)), \quad \forall x \in \mathcal{C}, \tag{9.70}$$

(since for $k \geq 3$ and large enough n, $k(n-1)/n(1-o(1)) > 1$). Note that by (5.1),

$$\begin{aligned} e^{-\{u(n-1)b_n/\varrho_{n,\mathcal{C}}(0)\}} &= \max\nolimits_{\{x,y\}\in G(\mathcal{C})} e^{-\{u(n-1)b_n/\min(w_n(y),w_n(x))\}} && (9.71)\\ &\leq \sum\nolimits_{\{x,y\}\in G(\mathcal{C})} e^{-u/\bar\gamma_n(\{x,y\})}, && (9.72)\end{aligned}$$

where we now set

$$\bar\gamma_n(\mathcal{C}') = \min_{x\in\mathcal{C}'} w_n(x)/(n-1)b_n, \quad \mathcal{C}' \in \mathcal{G}_2. \tag{9.73}$$

Combining these observations yields the bound

$$\bar\nu_n^{\circ,(k)}(u,\infty) \leq k\frac{a_n}{2^n} \sum\nolimits_{\mathcal{C}'\in\mathcal{G}_2} \sum\nolimits_{\mathcal{C}\in\mathcal{G}_k:\mathcal{C}'\subset\mathcal{C}} \prod\nolimits_{x\in\mathcal{C}} \chi_n(x) e^{-u/\bar\gamma_n(\mathcal{C}')}, \tag{9.74}$$

valid on $\Omega^\star$, for all but a finite number of indices n, and this in turns implies that

$$\mathbb{E}[\bar\nu_n^{\circ,(k)}(u,\infty)] \leq k(k-2)!n^{-(c_\star-1)(k-2)} v_n(u;\bar a_n, \bar b_n), \tag{9.75}$$

where $\bar a_n \equiv \sqrt{na_n}$, $\bar b_n \equiv (n-1)b_n$. Again one sees that these sequences (that differ but slightly from the choices made in (9.68)) satisfy the conditions of Lemma 9.7 with $\bar\varepsilon = \varepsilon/2$. Thus

$$\lim_{n\to\infty} \mathbb{E}\left[v_n(u;\bar a_n,\bar b_n)\right] = \nu^\dagger(u,\infty). \tag{9.76}$$

Since by assumption $c_\star > 2$ we may use (7.23) to sum (9.75) over k, which gives

$$\sum\nolimits_{3\leq k\leq k_n^\star} \mathbb{E}[\bar\nu_n^{\circ,(k)}(u,\infty)] \leq 4n^{-(c_\star-1)} \mathbb{E}\left[v_n(u;\bar a_n,\bar b_n)\right]. \tag{9.77}$$

Now set

$$\Delta_n(u) \equiv \sum\nolimits_{k\geq 3} \bar\nu_n^{\circ,(k)}(u,\infty) = \hat\nu_n^\circ(u,\infty) - \bar\nu_n^{\circ,(2)}(u,\infty) > 0. \tag{9.78}$$

By (9.77), using (2.26) to bound $r_n(\rho_n^\star)$, we obtain that under the assumptions of the lemma, for all $u > 0$,

$$0 \leq \mathbb{E}\Delta_n(u) \leq 4n^{-(c_\star-1)} \left(\mathbb{E}\left[v_n(u;\bar a_n,\bar b_n)\right] + e^{-\beta\sqrt{8\varepsilon n\log n}}\right), \tag{9.79}$$

and so $\lim_{n\to\infty} \mathbb{E}\Delta_n(u) = 0$. But this and (9.69) yield (9.57). The proof of Lemma 9.8 is complete. $\square$

Proof of Lemma 9.9. As in the proof of Lemma 9.8 we separate the contribution of $\bar\nu_n^{\circ,(2)}$ from those of $\bar\nu_n^{\circ,(k)}$, $k \geq 3$ (see (9.62) and (9.63) for their definitions). Namely, we write

$$\bar\nu_n^\circ(u,\infty) - \mathbb{E}[\bar\nu_n^\circ(u,\infty)] = \bar\nu_n^{\circ,(2)}(u,\infty) - \mathbb{E}[\bar\nu_n^{\circ,(2)}(u,\infty)] + \{\Delta_n(u) - \mathbb{E}[\Delta_n(u)]\} \tag{9.80}$$

where $\Delta_n(u)$ is defined in (9.78), and we take

$$\begin{aligned} V_{n,1}(u) &\equiv \mathbb{E}\left[v_n(u;\bar a_n,\bar b_n)\right] + e^{-\beta\sqrt{8\varepsilon n\log n}}, && (9.81)\\ V_{n,2}(u) &\equiv \mathbb{E}\left[v_n(u;a_n^+,b_n^+)\right], && (9.82)\end{aligned}$$

where $v_n(u; a_n^+, b_n^+)$ and $v_n(u; \bar{a}_n, \bar{b}_n)$ are as in (9.68) and (9.75), respectively. Eq. (9.59) then follows from (9.69) and (9.76). Note that $\Delta_n(u) > 0$. Thus, by (9.79),

$$\mathbb{P}\left(|\Delta_n(u) - \mathbb{E}[\Delta_n(u)]| \geq 4n^{-(c_\star - 1)} V_{n,1}(u)/L_1\right) \leq 2L_1, \quad \forall L_1 > 0. \tag{9.83}$$

Let us now establish that for all $L_2 \geq 0$ such that $na_nL_2/2^n = o(1)$,

$$\mathbb{P}\left(\left|\bar{\nu}_n^{\circ,(2)}(u,\infty) - \mathbb{E}[\bar{\nu}_n^{\circ,(2)}(u,\infty)]\right| \geq 4n^4\sqrt{na_nL_2V_{n,2}(2u)/2^n}\right) \leq 4n^5 e^{-L_2}. \tag{9.84}$$

Eq. (9.63) prompts us to set $\mathcal{S}_n \equiv \sum_{\mathcal{C}\in\mathcal{G}_2} X(\mathcal{C})$, where $X(\mathcal{C}) \equiv Y(\mathcal{C}) - \mathbb{E}Y(\mathcal{C})$ and

$$Y(\mathcal{C}) \equiv \prod_{x\in\mathcal{C}} \chi_n(x) \prod_{x'\in\partial\mathcal{C}} \overline{\chi}_n(x')\left(1 - \left[1 + \tfrac{\min_{x\in\mathcal{C}} w_n(x)}{(n-1)}\right]^{-1}\right)^{\lceil b_n u\rceil}, \quad \mathcal{C} \in \mathcal{G}_2. \tag{9.85}$$

Then

$$\mathbb{P}\left(\left|\bar{\nu}_n^{\circ,(2)}(u,\infty) - \mathbb{E}[\bar{\nu}_n^{\circ,(2)}(u,\infty)]\right| \geq \theta\right) = \mathbb{P}\left(|\mathcal{S}_n| \geq 2^{n-1}a_n^{-1}\theta\right). \tag{9.86}$$

To bound the last probability we again proceed as in the proof of Lemma 6.10 and, using (6.42), split $\mathcal{S}_n$ into $2n$ disjoint sums,

$$\mathcal{S}_n = \sum_{i=1}^{v_n}\sum_{j=1}^{n} \mathcal{S}_n^{j,i}, \quad \mathcal{S}_n^{j,i} \equiv \sum_{\mathcal{C}\in\mathcal{G}_2^{j,i}} X(\mathcal{C}). \tag{9.87}$$

Each $\mathcal{S}_n^{j,i}$ now is a sum of independent random variables, and can be controlled using Bennett's bound [9] for the tail behavior of sums of random variables, which we specialize as follows: if $(X(i),\, i \in \mathcal{I})$ is a family of i.i.d. centered random variables that satisfies $\max_i |X(i)| \leq 1$ then, setting $\tilde{B}^2 = \sum_{i\in\mathcal{I}} \mathbb{E}X^2(i)$, for all $B^2 \geq \tilde{B}^2$, for all $t < B^2/2$,

$$\mathbb{P}\left(\left|\sum_{i\in\mathcal{I}} X(i)\right| \geq t\right) \leq 2\exp\left\{-t^2/4B^2\right\}. \tag{9.88}$$

Since $|X(\mathcal{C})| \leq 1$ and $\sum_{\mathcal{C}\in\mathcal{G}_2^{j,i}} \mathbb{E}X^2(\mathcal{C}) \leq (na_n)^{-1}2^{n-1}V_{n,2}(2u) \equiv B^2$ (this follows from (9.64)-(9.68) and the rough bound $|\mathcal{G}_2^{j,i}| < |\mathcal{G}_2^j| = n^{-1}|\mathcal{G}_2|$), choosing $t^2 = 4L_2B^2$ yields

$$\mathbb{P}\left(|\mathcal{S}_n^{j\pm}| \geq \sqrt{2^{n-1}4L_2V_{n,2}(2u)/(na_n)}\right) \leq 2e^{-L_2}, \quad L_2 > 0. \tag{9.89}$$

In view of (9.67) this choice is permissible for all n large enough whenever $na_nL_2/2^n = o(1)$. Eq. (9.89) holds true for each $1 \leq j \leq n$ and $1 \leq i \leq v_n$, where $v_n \leq 2n^4$ (see (6.42)), and combined with (9.87) yields

$$\mathbb{P}\left(|\mathcal{S}_n| \geq 4n^5\sqrt{2^{n-1}L_2V_{n,2}(2u)/(na_n)}\right) \leq 4n^5e^{-L_2}, \tag{9.90}$$

which, by (9.86), is tantamount to (9.84). Combining (9.83) and (9.84) then yields (9.58) and concludes the proof of Lemma 9.9. □

9.3.2. *Convergence properties of $\bar{\sigma}_n^\circ$ and related functions.* We have:

Lemma 9.10. *Under the assumption and with the notation of Lemma 9.8,*

$$\lim_{n\to\infty} n\mathbb{E}[\bar{\sigma}_n^\circ(u,\infty)] = \lim_{n\to\infty} n\mathbb{E}[\bar{\sigma}_n^=(u,\infty)] = 2\nu^\dagger(2u,\infty), \quad \forall u > 0. \tag{9.91}$$

Lemma 9.11. *For all $L_1, L_3 > 0$ and $L_2 \geq 0$ such that $na_nL_2/2^n = o(1)$, for all $u > 0$,*

$$\mathbb{P}\left(|\bar{\sigma}_n^=(u,\infty) - \mathbb{E}[\bar{\sigma}_n^=(u,\infty)]| \geq \tilde{\phi}_n(u, L_1, L_2)\right) \leq 4n^5e^{-L_2} + 4L_1, \tag{9.92}$$

$$\mathbb{P}\left(|\bar{\sigma}_n^\circ(u,\infty) - \mathbb{E}[\bar{\sigma}_n^\circ(u,\infty)]| \geq \psi_n(u, L_1, L_2, L_3)\right) \leq 4n^5e^{-L_2} + 2L_1 + 2L_3, \tag{9.93}$$

where $\tilde{\phi}_n(u, L_1, L_2) \equiv 2n^4\sqrt{a_n L_2 W_{n,2}(2u)/2^n} + 4n^{-c_\star} W_{n,1}(u)/L_1$, $\psi_n(u, L_1, L_2, L_3) \equiv \tilde{\phi}_n(u, L_1, L_2) + 2^8 W^2_{n,3}(u)/(a_n L_3)$, *and where for each* n *and* $1 \leq i \leq 3$, $W_{n,i}(u)$ *are positive decreasing functions, while for each* $u > 0$, *under the assumptions of Lemma 9.8,*

$$\lim_{n\to\infty} W_{n,1}(u) = \lim_{n\to\infty} W_{n,2}(u) = 2\nu^\dagger(2u, \infty), \tag{9.94}$$

$$\lim_{n\to\infty} W^2_{n,3}(u) = [\nu^\dagger(u, \infty)]^2. \tag{9.95}$$

We prove Lemmata 9.10 and 9.11 simultaneously.

Proof of Lemma 9.10 and Lemma 9.11. Write $\bar{\sigma}^\circ_n(u, \infty) = \bar{\sigma}^=_n(u, \infty) + \bar{\sigma}^{\neq}_n(u, \infty)$ where

$$\bar{\sigma}^=_n(u, \infty) = \frac{a_n}{n2^n} \sum_{1\leq l\leq L^\star} \left[\sum_{x\in C^\star_{n,l}} Q^u_{n,l}(x)\right]^2, \tag{9.96}$$

$$\bar{\sigma}^{\neq}_n(u, \infty) = \frac{a_n}{n^2 2^n} \sum_{1\leq l\neq l'\leq L^\star} \sum_{x\in C^\star_{n,l}} \sum_{x'\in C^\star_{n,l'}} Q^u_{n,l}(x) Q^u_{n,l'}(x')|\partial x \cap \partial x'|. \tag{9.97}$$

Comparing (9.96) to (9.28), we see that $n\bar{\sigma}^=_n(u, \infty)$ differs from $\bar{\nu}^\circ_n(u, \infty)$ in that the term in square brackets is squared. However, examining the proof of 9.8 (see (9.63)-(9.66) and (9.70) -(9.74)) we also see that $n\bar{\sigma}^=_n(u, \infty)$ can be controlled in exactly the same way as $\bar{\nu}^\circ_n(u, \infty)$, substituting $|C^\star_{n,l}|^2 Q^{2u}_{n,l}(x)$ for $[\sum_{x\in C^\star_{n,l}} Q^u_{n,l}(x)]^2$. This yields

$$\lim_{n\to\infty} n\mathbb{E}[\bar{\sigma}^=_n(u, \infty)] = 2\nu^\dagger(2u, \infty), \quad \forall u > 0. \tag{9.98}$$

Similarly, (9.92) is a rerun of the proof of Lemma 9.9. Let us now establish that

$$\mathbb{E}[\bar{\sigma}^{\neq}_n(u, \infty)] \leq 2^8 a_n^{-1}[W_{n,3}(u)]^2 \tag{9.99}$$

for some positive decreasing function $W_{n,3}(u)$ of $u > 0$, that satisfies

$$\lim_{n\to\infty} W_{n,3}(u) = \nu^\dagger(u, \infty), \quad \forall u > 0. \tag{9.100}$$

For this write $\bar{\sigma}^{\neq}_n(u, \infty) = \sum_{2\leq k,k'\leq k^\star_n} \hat{\sigma}^{\neq,(k,k')}_n(u, \infty)$ where, with the notation of (9.62),

$$\hat{\sigma}^{\neq,(k,k')}_n(u, \infty) \equiv \frac{a_n}{n^2 2^n} \sum^{(1)}_{\mathcal{C},\mathcal{C}'} \phi(\mathcal{C}, \mathcal{C}') \sum^{(2)}_{x,x'} Q^u_{n,\mathcal{C}}(x) Q^u_{n,\mathcal{C}'}(x'). \tag{9.101}$$

Here the first sum, $\Sigma^{(1)}$, is over all $\mathcal{C} \in \mathcal{G}_k$ and $\mathcal{C}' \in \mathcal{G}_{k'}$ such that $\mathrm{dist}(\mathcal{C}, \mathcal{C}') = 2$, the second one, $\Sigma^{(2)}$, is over all $x \in \mathcal{C}$ and $x' \in \mathcal{C}'$ such that $\mathrm{dist}(x, x') = 2$, and $\phi(\mathcal{C}, \mathcal{C}') \equiv \prod_{y\in\mathcal{C}\cup\mathcal{C}'} \chi_n(y) \prod_{y'\in\partial\mathcal{C}\cup\partial\mathcal{C}'} \overline{\chi}_n(y')$. Thus $\mathcal{C} \cap \mathcal{C}' = \emptyset$, so that that $Q^u_{n,\mathcal{C}}(x)$ and $Q^u_{n,\mathcal{C}'}(x')$ are independent random variables for all $x \in \mathcal{C}$, $x' \in \mathcal{C}'$, and averaging out,

$$\mathbb{E} \sum_{2\leq k,k'\leq k^\star_n} \hat{\sigma}^{\neq,(k,k')}_n(u, \infty) \leq a_n^{-1}[W_{n,3}(u)]^2 s^{(3)}_n s^{(4)}_n, \tag{9.102}$$

where

$$W_{n,3}(u) \equiv \mathbb{E}\left[v_n(u; \bar{a}_n, \bar{b}_n)\right] + e^{-\beta\sqrt{8\varepsilon n\log n}} \tag{9.103}$$

for some sequences $\bar{a}_n$, $\bar{b}_n$ chosen as in Lemma 9.7, and where

$$s^{(i)}_n = \sum_{2\leq k\leq k^\star_n} k^i(k-1)!n^{-(c_\star-1)(k-2)}, \quad i \geq 1. \tag{9.104}$$

To see this, reason that there are at most $2^n(k-1)!n^{k-1}$ sets $\mathcal{C} \in \mathcal{G}_k$, that for each $\mathcal{C} \in \mathcal{G}_k$ there are at most $n^2k'(k'-1)!n^{k'-1}$ sets $\mathcal{C}' \in \mathcal{G}_{k'}$ such that $\mathrm{dist}(\mathcal{C}, \mathcal{C}') = 2$, that $\Sigma^{(2)}$ contains at most kk' terms, and that, proceeding as in (9.70)-(9.74) to bound the terms $Q^u_{n,\mathcal{C}}$ when $k > 2$, and proceeding as in (9.64)-(9.67) when $k = 2$, we have

$$\mathbb{E}\left[\phi(\mathcal{C}, \mathcal{C}') Q^u_{n,\mathcal{C}}(x) Q^u_{n,\mathcal{C}'}(x')\right] \leq (kk')^2(na_n)^{-2} n^{-c_\star[(k-2)+(k'-2)]}[W_{n,3}(u)]^2, \tag{9.105}$$

for some sequences $\bar{a}_n$, $\bar{b}_n$ for which the assumptions of Lemma 9.7 are verified, and all $k, k' \geq 2$. Now for $c_\star > 2$, by (7.23), $s_n^{(i)} \leq 2^i(1+2^{i+1}n^{-2(c_\star-1)})$. Inserting this in (9.102) and using Lemma 9.7 proves the claim (9.99)-(9.100). This immediately implies that

$$\lim_{n\to\infty} n\mathbb{E}[\bar{\sigma}_n^{\neq}(u,\infty)] = 0, \quad \forall u > 0, \tag{9.106}$$

and that, under the assumptions and with the notation of (9.99)-(9.100), for all $u > 0$,

$$\mathbb{P}\left(\left|\bar{\sigma}_n^{\neq}(u,\infty) - \mathbb{E}[\bar{\sigma}_n^{\neq}(u,\infty)]\right| \geq 2^8[W_{n,3}(u)]^2/(a_n L_3)\right) \leq 2L_3. \tag{9.107}$$

Lemma 9.10 now follows from (9.98) and (9.106), and (9.93) of Lemma 9.11 follows from(9.92), (9.100), and (9.107). ☐

We now prove Lemma 9.5.

Proof of Lemma 9.5. Let us establish first that for all $m \geq 1$, if $c_\star > 2$,

$$\mathbb{E}\big[I_{2,m}^{(3)}\big] \leq n^{-1}a_n^{-2}2^n[W_{n,3}(u)]^2 2^5(1+2^4 n^{-2(c_\star-1)})^2, \tag{9.108}$$

where $W_{n,3}(u) > 0$ is a decreasing function satisfying $\lim_{n\to\infty} W_{n,3}(u) = \nu^\dagger(u,\infty)$ for all $u > 0$. For this note that $I_{2,m}^{(3)}$ in (9.46) is very similar to the quantity $\bar{\sigma}_n^{\neq}(u,\infty)$ defined in (9.97). This prompts us to write

$$I_{2,m}^{(3)} = \textstyle\sum_{2\leq k,k'\leq k_n^\star} I_{2,m}^{(3),(k,k')} \tag{9.109}$$

where, for $\phi(\mathcal{C},\mathcal{C}')$ as in (9.101),

$$I_{2,m}^{(3),(k,k')} \equiv \textstyle\sum_{\mathcal{C},\mathcal{C}'}^{(1)} \phi(\mathcal{C},\mathcal{C}') \sum_{x,x'}^{(2)} \sum_{y,y'}^{(3)} Q_{n,l}^u(x) Q_{n,l'}^u(x') f_n^{\circ,m}(x,x';y,y'). \tag{9.110}$$

Here the first sum, $\Sigma^{(1)}$, is over all $\mathcal{C} \in \mathcal{G}_k$ and $\mathcal{C}' \in \mathcal{G}_{k'}$ such that $\mathcal{C} \cap \mathcal{C}' = \emptyset$, the second one, $\Sigma^{(2)}$, is over all $x \in \mathcal{C}$ and $x' \in \mathcal{C}'$, and the third one, $\Sigma^{(3)}$, is over all $y \in \partial\mathcal{C}$ and $y \in \partial\mathcal{C}'$. Since $\mathcal{C} \cap \mathcal{C}' = \emptyset$, $Q_{n,\mathcal{C}}^u(x)$ and $Q_{n,\mathcal{C}'}^u(x')$ are independent random variables for all $x \in \mathcal{C}$, $x' \in \mathcal{C}'$. Thus we see, using (9.105), that for all $k, k' \geq 2$,

$$\begin{aligned}&\mathbb{E}\big[I_{2,m}^{(3),(k,k')}\big]\\ &\leq (kk')^2(na_n)^{-2}n^{-c_\star[(k-2)+(k'-2)]}[W_{n,3}(u)]^2 \textstyle\sum_{\mathcal{C},\mathcal{C}'}^{(1)} \sum_{x,x'}^{(2)} \sum_{y,y'}^{(3)} f_n^{\circ,m}(x,x';y,y'),\end{aligned} \tag{9.111}$$

where $W_{n,3}(u)$ is given by (9.103) for some sequences $\bar{a}_n$, $\bar{b}_n$ for which the assumptions of Lemma 9.7 are verified – hence it has the properties claimed in the line below (9.108). To deal with the sums in (9.111), observe that given any $\mathcal{C} \in \mathcal{G}_k$, $x \in \mathcal{C}$, and $y \in \partial\mathcal{C}$,

$$\begin{aligned} f_n^{\circ,m}(x;y) &\equiv \textstyle\sum_{\mathcal{C}'\in\mathcal{G}_{k'}} \sum_{x'\in\mathcal{C}'} \sum_{y'\in\partial\mathcal{C}'} f_n^{\circ,m}(x,x';y,y') && (9.112)\\ &= p_n(x,y) \textstyle\sum_{y'\in\mathcal{V}_n^\circ} \sum_{\mathcal{C}'}^{(4)} \sum_{x'\in\mathcal{C}'} p_n(y',x') p_n^{\circ,m}(y,y') && (9.113)\end{aligned}$$

where the sum $\Sigma^{(4)}$ is over all $\mathcal{C}' \in \mathcal{G}_{k'}$ such that $\mathcal{C}' \cap \partial y' \neq \emptyset$. Indeed if $\mathcal{C}' \cap \partial y' = \emptyset$ then $p_n(y',x') = 0$ for all $x' \in \mathcal{C}'$. Now $\sum_{x'\in\mathcal{C}'} p_n(y',x') \leq 1$ while the number of terms in $\Sigma^{(4)}$ is at most $k'!n^{k'}$. Thus

$$f_n^{\circ,m}(x;y) \leq k'!n^{k'} p_n(x,y) \textstyle\sum_{y'\in\mathcal{V}_n^\circ} p_n^{\circ,m}(y,y') \leq k'!n^{k'} p_n(x,y), \tag{9.114}$$

From this we readily get

$$\textstyle\sum_{\mathcal{C},\mathcal{C}'}^{(1)} \sum_{x,x'}^{(2)} \sum_{y,y'}^{(3)} f_n^{\circ,m}(x,x';y,y') \leq (k-1)!n^{k-1}k'!n^{k'}, \tag{9.115}$$

and inserting this bound in (9.111) and (9.109) successively yields

$$\mathbb{E}\big[I_{2,m}^{(3)}\big] \leq n^{-1}a_n^{-2}2^n[W_{n,3}(u)]^2 s_n^{(2)} s_n^{(3)}, \tag{9.116}$$

where $s_n^{(i)}$ is defined in (9.104) and obeys $s_n^{(i)} \leq 2^i(1+2^{i+1}n^{-2(c_\star-1)})$ whenever $c_\star > 2$. Eq. (9.108) now immediately follows. Invoking (6.7) of Proposition 6.3 we get that on $\Omega^\star$, for all but a finite number of indices n, if $c_\star > 2$,

$$(a_n/|\mathcal{V}_n^\circ|)\sum_{m=1}^{\ell_n^\circ-1}\mathbb{E}\big[I_{2,m}^{(3)}\big] \leq 2^5(1+o(1))\frac{\ell_n^\circ}{na_n}[W_{n,3}(u)]^2. \tag{9.117}$$

The lemma now follows by a first order Tchebychev inequality. □

9.3.3. *Proof of Proposition 9.6.* The proof of Proposition 9.6 is now a mere formality. Recall that $c_\star > 2$ and that a_n obeys (9.52) for some $0 < \varepsilon < 1$. Choose $L_1 = n^{-1-(c_\star-2)/2}$ and $L_2 = 7\log n$ in Lemma 9.9. Then $na_nL_2/2^n = o(1)$, $\lim_{n\to\infty}\phi_n(u,L_1,L_2) \to 0$, and $\sum_n(2n^5e^{-L_2}+2L_1) < \infty$, so that by Lemma 9.8, Lemma 9.9, and Borel-Cantelli Lemma,

$$\lim_{n\to\infty}\bar\nu_n^\circ(u,\infty) = \nu^\dagger(u,\infty), \quad \mathbb{P}-\text{almost surely}, \tag{9.118}$$

for all $u > 0$. Because $\bar\nu_n^\circ(u,\infty)$ is a sequence of monotonic functions of $u > 0$ whose limit, $\nu^\dagger(u,\infty)$, is continuous, (9.118) entails the existence of a subset $\overline{\Omega}_1^{\mathsf{LLN}} \subset \Omega$ with the property that $\mathbb{P}(\overline{\Omega}_1^{\mathsf{LLN}}) = 1$, and such that on $\overline{\Omega}_1^{\mathsf{LLN}}$,

$$\lim_{n\to\infty}\bar\nu_n^\circ(u,\infty) = \nu^\dagger(u,\infty), \quad \forall u > 0. \tag{9.119}$$

We prove in the same way, using the monotonicity of $\bar\sigma_n^\circ(u,\infty)$, Lemma 9.10, and Lemma 9.11 (with L_1 and L_2 as above and $L_3 = n^{-2}$, so that $\lim_{n\to\infty}n\psi_n(u,L_1,L_2,L_3) = 0$ and $\sum_n(2n^5e^{-L_2}+2L_1+2L_3) < \infty$) that there exists a subset $\overline{\Omega}_2^{\mathsf{LLN}} \subset \Omega$ of full measure such that, on $\overline{\Omega}_2^{\mathsf{LLN}}$,

$$\lim_{n\to\infty}n\bar\sigma_n^\circ(u,\infty) = 2\nu^\dagger(2u,\infty), \quad \forall u > 0. \tag{9.120}$$

Taking $\overline{\Omega}^{\mathsf{LLN}} = \overline{\Omega}_1^{\mathsf{LLN}} \cap \overline{\Omega}_2^{\mathsf{LLN}}$ completes the proof of Proposition 9.6 .

9.4. **Conclusion of the proof of Theorem 3.4.** It suffices to prove that under the assumptions of Theorem 3.4, Conditions (A1), (A2), and (A3) of Theorem 9.1 are verified $\mathbb{P}$-almost surely when $\nu^\dagger$ in Condition (A1) is as in (1.23).

9.4.1. *Verification of Conditions (A1) and (A2).* It immediately follows Lemma 9.2, Theorem 9.3, Proposition 9.6, and Lemma 9.10 that under the assumptions therein, $\mathbb{P}$-almost surely, for all $u > 0$ and all $t > 0$,

$$\lim_{n\to\infty}\bar\nu_n^{J_n^\circ,t}(u,\infty) = t\nu^\dagger(u,\infty) \ \text{ in } P^\circ\text{-probability}, \tag{9.121}$$

$$\lim_{n\to\infty}\bar\sigma_n^{J_n^\circ,t}(u,\infty) = 0 \ \text{ in } P^\circ\text{-probability}. \tag{9.122}$$

Conditions (A1) and (A2) are thus satisfied $\mathbb{P}$-almost surely.

9.4.2. *Verification of Condition (A3).* This still requires a little work. Given $\epsilon > 0$, define

$$\eta_{n,k}(\epsilon) \equiv \frac{a_n}{2^n}\textstyle\sum_{1\leq l\leq L^\star}\mathbb{1}_{\{|C_{n,l}^\star|=k\}}\sum_{x\in C_{n,l}^\star}A_{n,l}(x), \quad k \geq 2, \tag{9.123}$$

where, given $x \in C_{n,l}^\star$, $1 \leq l \leq L^\star$,

$$A_{n,l}(x) \equiv E_x\big(\mathbb{1}_{\{b_n^{-1}T_{n,l}^\star\leq\epsilon\}}b_n^{-1}T_{n,l}^\star\big). \tag{9.124}$$

One readily sees that Condition (A3) will be verified $\mathbb{P}$-almost surely if

$$\lim_{\epsilon\downarrow 0}\limsup_{n\uparrow\infty}\textstyle\sum_{k\geq 2}\eta_{n,k}(\epsilon) = 0, \quad \mathbb{P}\text{-a.s.}. \tag{9.125}$$

Note that $\eta_{n,k}(\epsilon)$ is of the form (9.60) with $Q^u_{n,l}(x)$ replaced by $A_{n,l}(x)$ and hence, as in (9.62), may be written as

$$\eta_{n,k}(\epsilon) \equiv \frac{a_n}{2^n} \sum_{\mathcal{C}\in\mathcal{G}_k} \prod_{x\in\mathcal{C}} \chi_n(x) \prod_{x'\in\partial\mathcal{C}} \overline{\chi}_n(x') \sum_{x\in\mathcal{C}} A_{n,\mathcal{C}}(x), \tag{9.126}$$

where $A_{n,\mathcal{C}}(x)$ stands for $A_{n,l}(x)$ with $C^\star_{n,l} \equiv \mathcal{C}$. As in the proof of Lemma 9.8 we note that on $\Omega^\star$, $k^\star_n \le \frac{n}{(c_\star-2)\log n}$ for all large enough n, and treat the terms $k = 2$ and $3 \le k \le k^\star_n$ separately. Throughout the proof we set $\bar{a}_n = \sqrt{na_n}$, $\bar{b}_n = b_n(n-1)$, and define

$$\gamma_n(\mathcal{C}') = \min_{x\in\mathcal{C}} w_n(x)/\bar{b}_n, \quad \mathcal{C}' \in \mathcal{G}_2. \tag{9.127}$$

• ***The term*** $k = 2$**.** Let us establish that for all large enough n and small enough ϵ, the mean and variance of $\eta_{n,2}(\epsilon)$ obey

$$\begin{aligned} \mathbb{E}\eta_{n,2}(\epsilon) &\le 2^{-n/6} + 2\epsilon^{1-\left(2\alpha_c(\frac{\varepsilon}{2})+o(1)\right)}, &(9.128)\\ \mathbb{E}\left(\eta_{n,2}(\epsilon) - \mathbb{E}\eta_{n,2}(\epsilon)\right)^2 &\le 2\frac{a_n}{2^n}\left[2^{-n/6} + 4\epsilon^{2-\left(2\alpha_c(\frac{\varepsilon}{2})+o(1)\right)}\right]. &(9.129)\end{aligned}$$

We first prove (9.128). By Proposition 5.1, (i), and integration by parts, for all $x \in \mathcal{C}$,

$$\begin{aligned} A_{n,\mathcal{C}}(x) &\le \tfrac{1}{b_n}\textstyle\sum_{i=0}^{\lfloor b_n\epsilon\rfloor}\left(1 - \frac{1}{1+\min_{x\in\mathcal{C}} w_n(x)/(n-1)}\right)^i &(9.130)\\ &\le [1+o_{n,1}(1)]\varphi_\epsilon\left([1+o_{n,2}(1)]\min_{x\in\mathcal{C}} w_n(x)/\bar{b}_n\right) &(9.131)\end{aligned}$$

where $|o_{n,i}(1)| \le \mathcal{O}(r_n^{-1}(\rho^\star_n))$, $i = 1, 2$, and

$$\varphi_\epsilon(y) = y(1 - e^{-\epsilon/y}), \quad y \ge 0. \tag{9.132}$$

Plugging (9.131) in (9.126) yields

$$\mathbb{E}\eta_{n,2}(\epsilon) \le [1+o_{n,1}(1)]\mathbb{E}\left[\bar{a}_n^2\varphi_\epsilon\left([1+o_{n,2}(1)]\gamma_n(\mathcal{C})\right)\mathbb{1}_{\{\gamma_n(\mathcal{C})\ge r_n(\rho^\star_n)/\bar{b}_n\}}\right]. \tag{9.133}$$

Now for $\epsilon > r_n(\rho^\star_n)/\bar{b}_n$ split $\mathbb{1}_{\{\gamma_n(\mathcal{C})\ge r_n(\rho^\star_n)/\bar{b}_n\}}$ into $\mathbb{1}_{\{\gamma_n(\mathcal{C})\ge\epsilon\}} + \mathbb{1}_{\{r_n(\rho^\star_n)/\bar{b}_n\le\gamma_n(\mathcal{C})<\epsilon\}}$. On the one hand, observing that $\varphi_\epsilon(y) \le \epsilon$ for all $y > 0$, we have

$$\mathbb{E}\left[\bar{a}_n^2\varphi_\epsilon\left([1+o_{n,2}(1)]\gamma_n(\mathcal{C})\right)\mathbb{1}_{\{\gamma_n(\mathcal{C})\ge\epsilon\}}\right] \le \epsilon\bar{a}_n^2\mathbb{P}\left(\gamma_n(\mathcal{C})\ge\epsilon\right) = \epsilon^{1-2\left(\alpha_c(\frac{\varepsilon}{2})+o(1)\right)}, \tag{9.134}$$

where the last equality follows from Lemma 2.1 (i) of [25]. On the other hand $\varphi_\epsilon(y) \le y$ for all $y > 0$. This and integration by part yields, setting $F_n(v) = \bar{a}_n\mathbb{P}(w_n(x) \ge v\bar{b}_n)$,

$$\begin{aligned} &\mathbb{E}\left[\bar{a}_n^2\varphi_\epsilon\left([1+o_{n,2}(1)]\gamma_n(\mathcal{C})\right)\mathbb{1}_{\{r_n(\rho^\star_n)/\bar{b}_n\le\gamma_n(\mathcal{C})<\epsilon\}}\right]\\ &\qquad\qquad \le [1+o_{n,2}(1)]\textstyle\int_{r_n(\rho^\star_n)/\bar{b}_n}^{\epsilon} F_n^2(y)dy. &(9.135)\end{aligned}$$

Given $0 < \delta < 1$, split the domain of integration in (9.135) into $[r_n(\rho^\star_n)/\bar{b}_n, \bar{b}_n^{-\delta}] \cup [\bar{b}_n^{-\delta}, \epsilon]$. Using that $F_n^2(y) \le \bar{a}_n^2$ on the first domain, and using Lemma 2.1 (ii) of [25] on the 2nd,

$$\textstyle\int_{r_n(\rho^\star_n)/\bar{b}_n}^{\epsilon} F_n^2(y)dy \le \bar{a}_n^2\bar{b}_n^{-\delta} + \frac{1+o(1)}{1-2(1-\frac{\delta}{2})\alpha_n}\left(\frac{1}{1-\delta}\right)^2\epsilon^{1-2(1-\frac{\delta}{2})\alpha_n}, \tag{9.136}$$

where $0 \le \alpha_n = \alpha_c(\frac{\varepsilon}{2}) + o(1)$. By definition of $\bar{a}_n$, $\bar{b}_n$, (2.25), and the assumption that $\beta > 2\beta_c(\varepsilon/2)$, we get $\bar{a}_n^2\bar{b}_n^{-\delta} \le \exp\left\{n\left[\beta_c^2(\varepsilon/2)(1 - 2\delta(1+o(1)))\right]\right\}$. Hence, choosing $\delta = 2/3$, $\int_{r_n(\rho^\star_n)/\bar{b}_n}^{\epsilon} F_n^2(y)dy \le 2^{-n/6} + \frac{9}{1-[2\alpha_c(\varepsilon/2)+o(1)](2/3)}\epsilon^{1-\frac{2}{3}[2\alpha_c(\varepsilon/2)+o(1)]}$. Collecting our bounds we arrive at (9.128).

Turning to the variance we have

$$\mathbb{E}\left(\eta_{n,2}(\epsilon) - \mathbb{E}\eta_{n,2}(\epsilon)\right)^2 = \left(\frac{a_n}{2^n}\right)^2\mathbb{E}\left(\textstyle\sum_{\mathcal{C}\in\mathcal{G}_2}[Y_n(\mathcal{C}) - \mathbb{E}Y_n(\mathcal{C})]\right)^2, \tag{9.137}$$

where $Y_n(\mathcal{C}) \equiv \prod_{x\in\mathcal{C}} \chi_n(x) \prod_{x'\in\partial\mathcal{C}} \overline{\chi}_n(x') \sum_{x\in\mathcal{C}} A_{n,\mathcal{C}}(x)$, $\mathcal{C} \in \mathcal{G}_2$. Observing that $Y_n(\mathcal{C})$ and $Y_n(\mathcal{C}')$ are independent whenever $\mathcal{C} \neq \mathcal{C}'$ and $\partial\mathcal{C} \cap \partial\mathcal{C}' = \emptyset$, and that $Y_n(\mathcal{C})Y_n(\mathcal{C}') = 0$ whenever $\mathcal{C} \neq \mathcal{C}'$ and $\mathcal{C} \cap \mathcal{C}' \neq \emptyset$, we readily get that $\mathbb{E}\left(\eta_{n,2}(\epsilon) - \mathbb{E}\eta_{n,2}(\epsilon)\right)^2 \leq I_n^{=} + I_n^{\neq}$,

$$I_n^{=} \equiv \left(\frac{a_n}{2^n}\right)^2 \sum_{\mathcal{C}\in\mathcal{G}_2} \mathbb{E}[Y_n^2(\mathcal{C})], \tag{9.138}$$

$$I_n^{\neq} \equiv \left(\frac{a_n}{2^n}\right)^2 \sum^{(1)}_{\mathcal{C},\mathcal{C}'} (\mathbb{E}[Y_n(\mathcal{C})Y_n(\mathcal{C}')] - [\mathbb{E}Y_n(\mathcal{C})][\mathbb{E}Y_n(\mathcal{C}')]), \tag{9.139}$$

where, as in (9.101), the sum $\Sigma^{(1)}$ is over all $\mathcal{C} \in \mathcal{G}_2$ and $\mathcal{C}' \in \mathcal{G}_{2'}$ such that $\mathrm{dist}(\mathcal{C}, \mathcal{C}') = 2$. We bound (9.138) in just the same way as $\mathbb{E}\eta_{n,2}(\epsilon)$, namely, using (9.131) in (9.138) gives

$$I_n^{=} \leq 2\frac{a_n}{2^n}[1 + o_{n,1}(1)]\mathbb{E}\left[\bar{a}_n^2 \varphi_\epsilon^2\left([1 + o_{n,2}(1)]\gamma_n(\mathcal{C})\right) \mathbb{1}_{\{\gamma_n(\mathcal{C}) \geq r_n(\rho_n^\star)/\bar{b}_n\}}\right], \tag{9.140}$$

and proceeding as in (9.133)-(9.136) to evaluate (9.133), we obtain (9.129). To Bound $I_n^{\neq}$ note that

$$I_n^{\neq} = \left(\frac{a_n}{2^n}\right)^2 \sum^{(1)}_{\mathcal{C},\mathcal{C}'} \mathbb{E}[Z_n(\mathcal{C})]\mathbb{E}[Z_n(\mathcal{C}')]\Delta_n(\mathcal{C}, \mathcal{C}') \tag{9.141}$$

where $Z_n(\mathcal{C}) \equiv \prod_{x\in\mathcal{C}} \chi_n(x) \sum_{x\in\mathcal{C}} A_{n,\mathcal{C}}(x)$, and

$$\begin{aligned}\Delta_n(\mathcal{C}, \mathcal{C}') &\equiv \mathbb{E}[\textstyle\prod_{y\in\partial\mathcal{C}\cup\partial\mathcal{C}'} \overline{\chi}_n(y)] - \mathbb{E}[\prod_{x\in\partial\mathcal{C}} \overline{\chi}_n(x)]\mathbb{E}[\prod_{x'\in\partial\mathcal{C}'} \overline{\chi}_n(x')] \\ &= (1 - n^{-c_\star})^3(1 - (1 - n^{-c_\star})).\end{aligned} \tag{9.142}$$

Observing that the right hand side of (9.133) (and a fortiori the r.h.s. of (9.128)) is an upper bound on $\bar{a}_n^2\mathbb{E}[Z_n(\mathcal{C})]$, and that the sum $\Sigma^{(1)}$ contains at most $4!n^4 2^{n-1}$ terms, we obtain

$$I_n^{\neq} \leq 4!(1 - n^{-c_\star})^3 n^{2-c_\star} 2^{-n}\left(2^{-n/6} + 2\epsilon^{1-\left(2\alpha_c\left(\frac{\varepsilon}{2}\right)+o(1)\right)}\right)^2. \tag{9.143}$$

Combining (9.140) and (9.143) now yields (9.129).

Since $n^2 \sum_n a_n/2^n < \infty$ it follows from (9.128), Borel-Cantelli Lemma (through a second order Tchebychev inequality), and (9.129) that $\lim_{n\uparrow\infty} \eta_{n,2}(\epsilon) = 2\epsilon^{1-2\alpha_c(\varepsilon/2)}$ $\mathbb{P}$-almost surely, for all $\varepsilon > 0$. Observing that $\eta_{n,2}(\epsilon)$ is a monotonic function of ϵ, and arguing as in the proof of Proposition 9.6 (see (9.118)-(9.119)), we obtain that

$$\lim_{\epsilon\downarrow 0} \limsup_{n\uparrow\infty} \eta_{n,2}(\epsilon) = 0, \quad \mathbb{P}\text{- almost surely.} \tag{9.144}$$

• ***The terms*** $3 \leq k \leq k_n^\star$**.** Note that $\eta_n(\epsilon) - \eta_{n,2}(\epsilon) > 0$. Our strategy is to bound $\mathbb{E}(\eta_n(\epsilon) - \eta_{n,2}(\epsilon))$ from above and use a first order Tchebychev inequality to infer from it $\mathbb{P}$-a.s. convergence of $\eta_n(\epsilon) - \eta_{n,2}(\epsilon)$ to zero. Since

$$0 < \eta_n(\epsilon) - \eta_{n,2}(\epsilon) = \sum_{k=3}^{k_n^\star} \eta_{n,k}(\epsilon), \tag{9.145}$$

it suffices to bound each $\mathbb{E}\eta_{n,k}(\epsilon)$.

As in the proof of Lemma 9.8 we denote by $\varrho_{n,\mathcal{C}}(0)$ the quantity $\varrho_{n,l}(0)$ from (5.1) with $C_{n,l}^\star \equiv \mathcal{C}$. Similarly $T_{n,\mathcal{C}}^\star$ and $A_{n,\mathcal{C}}(x)$ stand for $T_{n,l}^\star$ and $A_{n,l}(x)$, respectively, with $C_{n,l}^\star \equiv \mathcal{C}$. Using Proposition 5.1, (ii) and proceeding as in (9.130)-(9.131), we get that on $\Omega^\star$, for all but a finite number of indices n, for all $x \in \mathcal{C}$,

$$\begin{aligned}A_{n,\mathcal{C}}(x) &\leq [1 + o(1)]\varphi_\epsilon\left([1 + o(1)]k\varrho_{n,\mathcal{C}}(0)/\bar{b}_n\right) &(9.146)\\ &\leq [1 + o(1)]\left(\epsilon 1_{\{k\varrho_{n,\mathcal{C}}(0)\geq\bar{b}_n\epsilon\}} + (k\varrho_{n,\mathcal{C}}(0)/\bar{b}_n)1_{\{k\varrho_{n,\mathcal{C}}(0)<\bar{b}_n\epsilon\}}\right) &(9.147)\\ &\equiv [1 + o(1)](A_{n,l}^{(1)}(x) + A_{n,l}^{(2)}(x)) &(9.148)\end{aligned}$$

where the second line follows from the bounds $\varphi_\epsilon(y) \leq \epsilon$ and $\varphi_\epsilon(y) \leq y$, both valid for all $y > 0$, and where the last line defines the terms $A_{n,l}^{(1)}(x)$ and $A_{n,l}^{(2)}(x)$. For $i = 1, 2$, let

$\eta^{(i)}_{n,k}(\epsilon)$ denote $\eta_{n,k}(\epsilon)$ with $A^{(i)}_{n,l}(x)$ substituted for $A_{n,\mathcal{C}}(x)$. To bound $\mathbb{E}[\eta^{(1)}_{n,k}(\epsilon)]$ observe that by (5.1) and (9.73),

$$\{k\varrho_{n,\mathcal{C}}(0) \geq \bar{b}_n\epsilon\} \subseteq \cup_{\{x,y\}\in G(\mathcal{C})}\{\bar{\gamma}_n(\{x,y\}) > \epsilon/kn\}. \tag{9.149}$$

Thus, by (9.126),

$$\eta^{(1)}_{n,k}(\epsilon) \leq \epsilon k\frac{a_n}{2^n}\sum_{\mathcal{C}'\in\mathcal{G}_2}\sum_{\mathcal{C}\in\mathcal{G}_k:\mathcal{C}'\subset\mathcal{C}}\prod_{x\in\mathcal{C}}\chi_n(x)\mathbb{1}_{\{\bar{\gamma}_n(\mathcal{C}')>\epsilon/kn\}}, \tag{9.150}$$

and averaging out,

$$\mathbb{E}[\eta^{(1)}_{n,k}(\epsilon)] \leq \epsilon k(k-2)!n^{-(c_\star-1)(k-2)}\left[\bar{a}_n\mathbb{P}(w_n(0) > \bar{b}_n\epsilon/kn)\right]^2, \tag{9.151}$$

where $\bar{a}_n$, $\bar{b}_n$ are as before (see the line below (9.75)). A simple Gaussian tail estimate gives $\left[\bar{a}_n\mathbb{P}(w_n(0) > \bar{b}_n\epsilon/n)\right]^2 \leq (kn/\epsilon)^{2\alpha(\varepsilon/2)(1+o(1))}$, and so

$$\mathbb{E}[\eta^{(1)}_{n,k}(\epsilon)] \leq \epsilon k(k-2)!n^{-(c_\star-1)(k-2)}(kn/\epsilon)^{2\alpha(\varepsilon/2)(1+o(1))}. \tag{9.152}$$

To bound $\mathbb{E}[\eta^{(2)}_{n,k}(\epsilon)]$ note that by (5.1)

$$A^{(2)}_{n,\mathcal{C}}(x) \leq \sum_{\{x,y\}\in G(\mathcal{C})}k\gamma_n(\{x,y\})\mathbb{1}_{\{k\gamma_n(\{x,y\})\leq\epsilon\}}. \tag{9.153}$$

Inserting this in (9.126) and proceeding as for $\mathbb{E}[\eta^{(1)}_{n,k}(\epsilon)]$, we get that

$$\mathbb{E}[\eta^{(2)}_{n,k}(\epsilon)] \leq k(k-2)!n^{-(c_\star-1)(k-2)}\mathbb{E}\left[\bar{a}_n^2\gamma_n(\mathcal{C}')\mathbb{1}_{\{\gamma_n(\mathcal{C}')\leq\epsilon/k\}}\right]. \tag{9.154}$$

An expectation similar to that appearing in (9.154) was estimated in (9.135). Proceeding as we did in (9.135)-(9.136) to reach (9.128), we obtain

$$\mathbb{E}[\eta^{(2)}_{n,k}(\epsilon)] \leq k(k-2)!n^{-(c_\star-1)(k-2)}\left[k2^{-n/6} + k2\epsilon^{1-\left(2\alpha_c(\frac{\varepsilon}{2})+o(1)\right)}\right]. \tag{9.155}$$

Finally, by (9.148), (9.152), and (9.155),

$$\mathbb{E}[\eta_{n,k}(\epsilon)] \leq \frac{k(k-2)!}{n^{(c_\star-1)(k-2)}}\left[\epsilon(nk/\epsilon)^{\left(2\alpha_c(\frac{\varepsilon}{2})+o(1)\right)} + k2^{-n/6} + k2\epsilon^{1-\left(2\alpha_c(\frac{\varepsilon}{2})+o(1)\right)}\right], \tag{9.156}$$

and using (7.23) to perform the sum over k, we arrive at

$$\mathbb{E}[\textstyle\sum_{k=3}^{k_n^\star}\eta_{n,k}(\epsilon)] \leq \frac{c_0}{n^{(c_\star-2)}}(\epsilon/n)^{1-\left(2\alpha_c(\frac{\varepsilon}{2})+o(1)\right)} + 2^{-n/6} \tag{9.157}$$

for some constant $0 < c_0 \equiv c_0(\beta,\varepsilon) < \infty$. Since by assumption $c_\star > 3$ and $1 - \left(2\alpha_c(\frac{\varepsilon}{2}) + o(1)\right) > 0$ for all n large enough, the r.h.s. of (9.157) is summabe in n. Combining this and (9.145) yields that $0 < \eta_n(\epsilon) - \eta_{n,2}(\epsilon) \to 0$ $\mathbb{P}$-a.s., for all $\varepsilon > 0$. Since each $\eta_{n,k}(\epsilon)$ in (9.126) is a monotonic function of ϵ, so is $\eta_n(\epsilon) - \eta_{n,2}(\epsilon)$, and thus, arguing again as in the proof of Proposition 9.6,

$$\lim_{\epsilon\downarrow 0}\limsup_{n\uparrow\infty}(\eta_n(\epsilon) - \eta_{n,2}(\epsilon)) = 0, \quad \mathbb{P}\text{-a.s.}. \tag{9.158}$$

Since (9.144) and (9.158) imply (9.125), Condition (A3) is verified $\mathbb{P}$-almost surely under the assumptions of Theorem 9.1.

Having established that all three conditions of Theorem 9.1 are satisfied $\mathbb{P}$-almost surely with $\nu^\dagger$ given by (1.23), the proof of Theorem 3.4 is done.

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